Nat. Volatiles & Essent. Oils, 2021; 8(4): 10049-10054

A Generalized Fixed Point Theorems On Multi Valued Mappings In B-Metric Spaces

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Abstract

In this paper ,we prove a fixed point theorem and a common fixed point theorem for generalize multi valued mappings in complete b-metric spaces.

Keywords: Fixed point ,b metrics space, cauchy,contraction

Introduction and Preliminaries

Nonlinear analysis is oneofthe most important areas of mathematical research.The fixed point theory i s a vital part of the study of physical sciences,computing sciences, and engineering.A famous fixed point theorem for contractive mapping in complete metric spaces was made by Stefanin 1922 Later, Czerwik (1993, 1998) has come up with b-metrics which generalized usual metric spaces. Alikhani et al. 2013; Boriceanu 2009;Kiziltunc and Mehemet 2013) have presented many results regarding generalized weak contractive multifunctions and b-metric spaces The following definitions will be needed in the sequel:

Definition 1([8])

Let X and Y be nonempty sets. T is said to be multi-valued mapping from X to Y

if T is a function for X to the power set of Y. we denote a multivalued map

by:

$T:X \to 2^{y}$

Definition 2([8])

A point of $x_0 \in X$ is said to be a fixed point of the multivalued mapping T if $x_0 \in Tx_0$.

Example 3([12])Every single valued mapping can be viewed as a multi-valued mapping. Let $f : X \to Y$ be a single valued mapping. Define $T:X \to 2^Y$ by $Tx = \{f(x)\}$. Note that T is a multi-valued mapping iff for each $x \in X$, $TX \subseteq Y$. Unless otherwise stated we always assume Tx is non-empty for each $x, y \in X$.

Definition 4([3])

Let (X, d) be a metric space. A map $T:X \to X$ is called contraction if there exists $0 \le \lambda < 1$ such that $d(Tx, Ty) \le \lambda d(x, y)$, for all $x, y \in X$.

Definition 5([8])

Let (X, d) be a metric space. We define the Hausdorff metric on CB(X) induced by d. That is

$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y,A)\}$

for all $A, B \in CB(X)$, where CB(X) denotes the family of all nonempty closed and bounded subsets of X and $d(x, B) = \inf\{d(x, b) : b \in B\}$, for all $x \in X$.

Definition 6([8])

Let (X, d) be a metric space. A map $T:X \to CB(X)$ is said to be multi valued contraction if there exists $0 \le \lambda < 1$ such that $H(Tx, Ty) \le \lambda d(x, y)$, for all $x, y \in X$.

Lemma 7([8])

If $A, B \in CB(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that

$d(a, b) \le H(A, B) + \varepsilon$.

Definition 8([2,7,10,11]) Let X be a nonempty set and let $s \ge 1$ be a given real number. A function $d:X \times X \to R+$ is called a b-metric provide that, for all $x, y, z \in X$,

1. $d(x, y) = 0$ if and only if $x = y$

2. $d(x, y) = d(y, x)$

3. $d(x, z) \le s[d(x, y) + d(y, z)]$.

A pair(X, d) is called a b-metric space.

Example 9([4])

The space $l_p(0 < p < 1)$, $l_p = \{(x_n: \sum_{n=1}^{\infty} | x_n|^p < \infty\}$, together with the function

$d: l_p \times l_p \to R^+$.

Example 10([4]) The space $L_p(0 < p < 1)$ for all real function x(t), t ∈ [0, 1] such that $\int_0^1 |x(t)|^p dt < \infty$, is b-metric space if we take $d(x, y) = (\int_0^1 |x(t) - y(t)|^p dt)^{\frac{1}{p}}$

Example 11([2])

Let X = {0, 1, 2,3,4} and d(2, 0) = d(0, 2) =d(2, 4) = m ≥ 2, d(0, 1) = d(1, 2) = d(0, 1) = d(2, 1) =d(2, 3)=d(3, 4) = 1 and d(0, 0) = d(1, 1) = d(2, 2) = 0. Then $d(x, y) \le \frac{m}{2} [d(x, z) + d(z, y)]$ for all x, y, z ∈ X. If m > 2,the ordinary triangle inequality does not hold.

Definition 12([4])

Let (X, d) be a b-metric space. Then a sequence (x_n) in X is called Cauchy sequence if and only if for all ε> 0 there exists n(ε) ∈ N such that for each m, n ≥ n(ε) we have $d(x_n, x_m)<\varepsilon$.

Definition 13([4]) Let be a (X, d) b-metric space. Then a sequence (x_n) in X is called convergent sequence if and only if there exists x ∈ X such that for all ε> 0 there exists n(ε) ∈ N such that for all n ≥ n(ε) we have$d(x_n, x)<\varepsilon$.

In this case we write $\lim_{n\to\infty} x_n = x$

MAIN RESULTS:

Definition 14

Let (Z, d) be a b-metric space with constant s ≥ 1. A map T: Z → CB(Z) is said to be multi valued generalized contraction if

$$H(Tx ,Ty) \le s_1 d(x,Ty)+s_2 d(y,Tx)+s_3\frac{d(x,Tx)}{1+d(y,Tx)}+s_4\frac{d(Tx,Ty)[1+d(Tx,Ty)]}{1+d(Ty,y)}$$

for all $x, y \in Z \geq 0$ and $s_i \geq 0$, $i = 1,2,3,4$ with $2s_1+s_2+s_3+s_4 < 1$. (1)

Theorem 15: Let (Z,d) be the complete b-metric space with constant s≥1.Let T:Z→ CB(Z) be a multivalued generalized contractionmapping.Then T has a unique fixed points.

Proof:

Let $\zeta_1 \in T\zeta_0$,$\zeta_2 \in T\zeta_1$ such that $d(\zeta_1 ,\zeta_2) \le H(T\zeta_0 ,T\zeta_1)+(s_1 +s_3)$

Now,

$H(T\zeta_0, T\zeta_1) \leq s_1 d(\zeta_0, T\zeta_1) + s_2 d(\zeta_1, T\zeta_0) + s_3 \frac{d(\zeta_0, T\zeta_0)}{1+d(\zeta_1, T\zeta_0)}$

$+ s_4 \frac{d(T\zeta_0, T\zeta_1)[1+d(T\zeta_1, T\zeta_0)]}{1+d(T\zeta_1, \zeta_1)} + (s_1+s_3)$

$\leq s_1 d(\zeta_0, \zeta_1) + s_3 \frac{d(\zeta_1, T\zeta_0)}{1+d(\zeta_1, T\zeta_0)} + s_4 \frac{d(\zeta_1, \zeta_2)[1+d(\zeta_2, \zeta_1)}{[1+d(\zeta_2, \zeta_1)]} + (s_1+s_3)$

$d(\zeta_1, \zeta_2) \leq s_1 d(\zeta_0, \zeta_2) + s_3 d(\zeta_0, \zeta_1) + s_4 d(\zeta_1, \zeta_2) + (s_1+s_3)$

$d(\zeta_1, \zeta_2) \leq s_1 d(\zeta_0, \zeta_1) + s_1 d(\zeta_1, \zeta_2) + s_3 d(\zeta_0, \zeta_1) + s_4 d(\zeta_1, \zeta_2) + (s_1+s_3)$

$d(\zeta_1, \zeta_2) - s_1 d(\zeta_1, \zeta_2) - s_4 d(\zeta_1, \zeta_2) \leq s_1 d(\zeta_0, \zeta_1) + s_3 d(\zeta_0, \zeta_1) + (s_1+s_3)$

$(1-s_1-s_4) d(\zeta_1, \zeta_2) \leq s_1 d(\zeta_0, \zeta_1) + s_3 d(\zeta_0, \zeta_1) + (s_1+s_3)$

$d(\zeta_1, \zeta_2) \leq \frac{(s_1+s_3)}{1-s_1-s_4} d(\zeta_0, \zeta_1) + \frac{(s_1+s_3)}{1-s_1-s_4}$

By lemma 7, $\zeta_3 \in T\zeta_2$

$d(\zeta_2, \zeta_3) \leq H(T\zeta_1, T\zeta_2) + \frac{(s_1+s_3)^2}{1-s_1-s_4}$

$d(\zeta_2, \zeta_3) \leq H(T\zeta_1, T\zeta_2) + \frac{(s_1+s_3)^2}{1-s_1-s_4}$

$\leq s_1 d(\zeta_1, \zeta_3) + s_2 d(\zeta_2, \zeta_2)) + s_3 d(\zeta_1, \zeta_2) + s_4 d(\zeta_2, \zeta_3) + (s_1+s_3)^2$

$d(\zeta_2, \zeta_3) - s_4 d(\zeta_2, \zeta_3) \leq s_1 d(\zeta_1, \zeta_2) + s_1 d(\zeta_2, \zeta_3) + s_2 d(\zeta_2, \zeta_2) + s_3 d(\zeta_1, \zeta_2) + (s_1+s_3)^2$

$d(\zeta_2, \zeta_3) - s_4 d(\zeta_2, \zeta_3) - s_1 d(\zeta_2, \zeta_3) \leq s_1 d(\zeta_1, \zeta_2) + s_3 d(\zeta_1, \zeta_2) + (s_1+s_3)^2$

$$(1 - s_1 - s_4) d(\zeta_1, \zeta_3) \leq (s_1 + s_3) d(\zeta_1, \zeta_2)(s_1 + s_3)^2$$

$$d(\zeta_2, \zeta_3) \leq \left(\frac{(s_1 + s_3)}{1 - s_1 - s_4}\right)^2 d(\zeta_1, \zeta_2) + 2\left(\frac{(s_1 + s_3)}{1 - s_1 - s_4}\right)^2$$

As we proceed, the sequence $\{\zeta_n\}$ is derived by induction in such a way that

$\zeta_n \in T\zeta_{n-1}, \zeta_{n+1} \in T\zeta_n$ such that,

$d(\zeta_n, \zeta_{n+1}) \leq \frac{(s_1+s_3)}{1-s1-s4} d(\zeta_{n-1}, \zeta_n) + \left(\frac{(s_1+s_3)}{1-s_1-s_4}\right)^n$ for all $n \in \mathrm{N}$

Let $C = \frac{(s_1+s_3)}{1-s1-s4}$, where C is constant

$d(\zeta_n, \zeta_{n+1}) \leq C\, d(\zeta_{n-1}, \zeta_n) + C^n$

$\leq C[Cd((\zeta_{n-2}, \zeta_{n-1}) + C^{n-1})] + C^n$

$\leq C^2 d(\zeta_{n-2}, \zeta_{n-1}) + CC^{n-1} + C^n$

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$d(\zeta_n, \zeta_{n+1}) \leq C^n d(\zeta_0, \zeta_1) + n\, C^n$

Since $C < 1$, $\sum C_n$ and $\sum nC_n$ have same radius of convergence. Then, $\{\zeta_n\}$ is a Cauchy sequence. But (Z, d) is a complete b-metric space, it follows that

$\{\zeta_n\}$ is convergent.

$\alpha = \lim_{n\to\infty} \zeta_n$

Now,

$d(\alpha, T\alpha) \leq s\ d(\alpha, \zeta_{n+1}) + d(\zeta_{n+1}, T\alpha)$

$d(\alpha, T\alpha) \leq s\ d(\alpha, \zeta_{n+1}) + d(T\zeta_n, T\alpha)$

using (1), we obtain

$d(\alpha, T\alpha) \leq s[d(\alpha, \zeta_{n+1})] + s\, [\, s_1 d(\zeta_n, T\zeta_n) + s_2 d(\alpha, T\alpha) + s_3 d(\zeta_n, T\alpha) + s_4 d(\alpha, T\zeta_n) + s_5 d(\zeta_n, \alpha) + s_6 d(\zeta_n, \alpha)\,]$

As $n \to \infty$,

$d(\alpha, T\alpha) \leq s\ [s_2 d(\alpha, T\alpha) + s_3 d(\alpha, T\alpha)]$

$(1 - (s_2 + s_3))\, d(\alpha, T\alpha) \leq 0$

The above inequality is true unless $d(\alpha, T\alpha) = 0$.

Thus, $T\alpha = \alpha$.

Now we show that α is the unique fixed point of T.

Assume that β is another fixed point of T.

Then we have $T\beta = \beta$ and

$d(\alpha, \beta) = d(T\alpha, T\beta) \leq s\ d(\alpha, T\beta) + d(\beta, T\alpha)$

we obtain,

$d(\alpha, \beta) \leq 2sd(\alpha, \beta)$.

This implies that $\alpha = \beta$.

This completes the proof.

Conclusions: In this paper, we have defined some new contractive mappings in b-metric spaces. Furthermore, we proved the existence and uniqueness of fixed points results for multivalued mappings in b-metricspaces.And also generalize existing contraction condition in the current literature.

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