Nat. Volatiles & Essent. Oils, 2021; 8(6): 3283-3299

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Natural Volatiles &
Essential Oils

Solution Of Linear Systems By Using LU Factorization In Matlab

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Abstract

Includes paper on the definition of a system of linear and nonlinear way of solving linear and we talked about aspecial case of the system $Ax = b$ inthis case is $b = 0$. And we also talked about way to solve the system $Ax = b$ (method of direct and indirect).And to talkeda bout algorithm for LU way to solve the system and your comparing between method by hand.

Key words (linear system, matrices, concept of factorization, method solution of linear system)

1.Introduction

Systems of linear equations are at the heart of linear algebra, and they introduce some of the most important concepts in a straightforward and tangible manner. A systematic method for solving systems of linear equations is presented in some sections.

Throughout the essay, this algorithm will be utilized for computations. Demonstrate how a system of linear equations can be represented by a vector equation and a matrix equation.

Problems requiring linear combinations of vectors will be reduced to queries about systems of linear equations as a result of this equivalence. The fundamental concepts of spanning and linear independence, will play an essential role throughout the text as we explore the beauty and power of linear algebra. The importance of linear algebra in applications has grown in lockstep with the development in computing capacity, with each new generation of hardware and software prompting a demand for even more capabilities. Through the rapid rise of parallel processing and large-scale computations, computer science has become inextricably intertwined with linear algebra. Scientists and technologists are today working on issues that were unimaginable only a few decades ago. In many scientific and business disciplines today, linear algebra has more potential benefit for students than any other college mathematics topic. The information in this publication lays the groundwork for future research in a variety of fascinating fields. Here are a few options; more will be discussed later. Oil exploration.

Every day, a ship's computers calculate thousands of independent systems of linear equations while searching for offshore oil resources. Geophones attached to mile-long cables behind the ship measure the waves as they bounce off subsurface rocks.

2.Systems of Linear Equation (Linear System)

In mathematic we define a linear equation by the form:

$$a_1x_1 + a_2x_2 + a_3x_3 + ...+ a_nx_n = b \qquad (1)$$

Where and are real or complex numbers. For examples:

$$2x_1 + x_2 - x_3 = 2 \qquad (2)$$

$$4x_1 - 5x_2 = x_1x_2 \qquad (3)$$

$$x_2 = 2\sqrt{x_1} - 5 \qquad (4)$$

Equation (3) and (4) are not liner because of the presence x_1x_2 in the second and

$\sqrt{x_1}$

in the there'd equation.

A collection of one or more linear equation involving in some variables $x_1, ...x_n$

Said to be system of linear equations for example:

$$2x_1 - x_2 + 5x_3 = 8$$

$$x_1 - 4x_3 = -7$$

(2.1) The matrix equation

If is matrix has n rows and m columns and , $a_1, ..., a_n \in R$, $x \in R^n$ then the product of and denoted by , is the linear combination of the columns of using the corresponding entries in as weights that is :

$$AX = (a_1 \ a_2 .... a_n)\begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} = a_1x_1 + a_2x_2 + a_3x_3 + ...+ a_nx_n \qquad (5)$$

Since the number of columns in A must equal the number of entries in , is defined.

(2.2) Homogeneous Linear Systems

A system of linear equations is said to be homogeneous if it can be written in the form , where A is an matrix and 0 is the zero vector in R^n.

Such a system always has at least one solution, namely ,x = 0 (the zero vector in R^n . The trivial solution is the name given to this zero solution. For

a given equation , the important question is whether there exists a nontrivial solution, that is, a nonzero vector x that satisfies .

The homogeneous equation has a nontrivial solution if and only if the equation has at least one free variable.

3. Solution of Linear System

(3.1) Solving the System $AX = 0$

We define the method of solution the system by next example**:**

(3.2) Example

Determine if there is a nontrivial solution to the following homogeneous system. Then describe the solution set.

$3x_1 + 5x_2 - 4x_3 = 0$

$-3x_1 - 2x_2 + 4x_3 = 0$

$6x_1 + x_2 - 8x_3 = 0$

Solution:

Let A be the matrix of coefficients of the system and row reduce the augmented matrix to echelon form:

$$\begin{pmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

 has nontrivial solutions (one for each choice of) since is a free variable. To describe the solution set, continue the row reduction of to reduced echelon form:

, since

$$\begin{pmatrix} 1 & 0 & \frac{-4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 - \frac{4}{3}x_3 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

Solve for the basic variables and , and obtain , ,withfree. As a vector, the general solution of Ax = 0has the form:

$$X = \begin{pmatrix} \frac{3}{4} \\ 0 \\ 1 \end{pmatrix}$$

(3.3) Solving the system

In linear algebra, solution sets of linear systems are fundamental objects to study. They will appear later in several different contexts.

(3.4) The inverse of a matrix

A matrix $A_{n\times m}$ is said to be invertible if there is an matrix such that and where ; then is identity matrix. In this case, is an inverse of . In fact, is uniquely deter If mined by ,because if B were another inverse of , then This unique inverse is denoted by , so that . A Singular matrix are invertible matrices that are not invertible sometimes., and an
invertible matrix is called a nonsingular matrix.

(3.5) Determinate of matrices

Let be square matrix of order , then the number is called determinate of the matrix .

(i) Determinate of matrix , the matrix has inverse if the determinant is not zero. let , if ,then is not invertible and is called determinate of , and denoted by .

(ii) Determinate of matrix .Let , then $B=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

. $|B|=a\begin{vmatrix} e & f \\ h & i \end{vmatrix}-b\begin{vmatrix} d & f \\ g & i \end{vmatrix}+c\begin{vmatrix} d & e \\ g & h \end{vmatrix}=a(ei-fh)-b(di-gf)+c(dh-ge)$

(3.6) Note

If , $|B|=0$ then has not inverse and called singular matrix.

(3.7) Example

Find the inverse of $A=\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$

Solution

Since , $\det=(3.6)-(4.5)=-2\neq 0$ is invertible, and $A^{-1}=\frac{-1}{2}\begin{pmatrix} 6 & -5 \\ -4 & 3 \end{pmatrix}$ Invertible matrices are indispensable in linear algebra mainly for algebraic calculations and formula derivations, as in the next theorem. There are also occasions when an inverse matrix provides insight into a mathematical model of a real-life situation, as in Example, below.

(3.8) Theorem (Unique Solution)

If is an invertible $n\times n$ matrix, then for each , $B\in R^{n}$ the equation has the unique $AX=B$ solution is . $X=A^{-1}B$

Proof:

Take any . A solution exists because if $A^{-1}B$ is substituted for , then $AX = A(A^{-1}B) = (A\,A^{-1})B = I\,B = B$ .So is a solution. To prove that the solution is unique, show that if is any solution, then ; in fact, must be indeed, if , we can multiply both sides by and obtain; . $A^{-1}A\,U = A^{-1}B, U = A^{-1}B$

4.Types of matrices

(4.1) (Square Matrix)
A square matrix has the same number of rows and columns as the number of columns.

(4.2) (Diagonal Matrix)
A square matrix $A = (a_{i,j})_{n*n}$ called a diagonal matrix if each of its non-diagonal element is zero. That is if $i \neq j$, and at least one element . $a_{ij} \neq 0$

(4.3) (Identity Matrix)
A diagonal matrix whose diagonal elements are equal to 1 is called identity matrix and denoted by .

That is . $\begin{pmatrix} 0, i \neq j \\ 1, i = j \end{pmatrix}$

(4.4) (Upper Triangular Matrix)

A square matrix said to be a upper triangular matrix if . $a_{ij} = 0, i > j$

(4.5) (Lower Triangular Matrix)

A square matrix said to be a Lower triangular matrix if . $a_{i\,j} = 0, i < j$

(4.6) Note:
If is a triangular matrix, then is the product of the entries on the main diagonal of .

(4.7) (Symmetric Matrix)

square matrix $A = (a_{i\,j})_{n \times n}$ is said to be a symmetric if . $a_{i\,j} = a_{j\,i}, \forall i, j$ **(4.8) (Column Vector)**

A column vector or column matrix is a matrix with only one column.

(4.9) (Elementary Matrix)
An elementary matrix is one that is produced from an identity matrix by conducting a single elementary row operation. The three types of elementary matrices are demonstrated in the following example.

(4.10) Example

Let: $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, calculate

, and describe how these products can be obtained by elementary row operation on .

Solution

$$. \; E_1 A = \begin{pmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{pmatrix}, \; E_2 A = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}, \; E_3 A = \begin{pmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{pmatrix}$$

−4 times row 1 of to row 3 produces , (This is a row re- placement operation.) An interchange of rows 1 and 2 of produces , and multiplication of row 3 of by 5 produces .**(4.11) Definition**

An indexed set of vectors $v_1, \ldots\ldots, v_p$ is said to be linearly independent if the vector equatio $c_1 v_1 + \ldots\ldots + c_p v_p = 0$ n has only the trivial solution $c_1 = c_2 = \ldots.. = c_p = 0$ .If there is ,Then the set is said to be linear dependent.

(4.12) (Column Space)

A set of all linear combination of the column of , is called column space.

(4.13) (Rank of Matrix)

rank of a matrix , denoted by rank , is the dimension of the column space of .**(4.14) (Null Space of Matrix)**

The null space of a matrix is the set of all solutions of the homogeneous equation .

(4.15) (Basis of Matrix)

A basis for a subspace of is a linearly independent set in that spans .

(4.16) (The Pivot Column of Matrix)

The pivot columns of a matrix form a basis for the column space of .

(4.17) (Dimensional of Matrix)

If is spanned by a finite set, then is said to be finite-dimensional, and the dimension of , written as , is the number of vectors in a basis for . The dimension of the zero vector space0 is defined to be zero. If is not spanned by a finite set, then is said to be infinite-dimensional

(4.18) (The Row Space of Matrix)

The set of all linear combination of the rows of ,and denoted by .

(4.19) Theorem

If two matrices and are row equivalent, then their row space the same.

If in echelon form the non-zero rows of form a basis for the row space of as well as the form .

(4.20) Example

Find bases for the row space, the null space of the matrix:

$$A = \begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix} \quad A \sim B = \begin{pmatrix} 1 & 3 & -5 & 0 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution:

$$. \; Rwo\ (A) = [(1,3,-5,1,5),(0,1,-2,2,-7),(0,0,0,-4,20)]$$

$$, \; Col\ (A) = \left\{ \begin{pmatrix} -2 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \\ 11 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 7 \\ 5 \end{pmatrix} \right\}$$ to find ,we have to find the reduced echelon form of .

, $A \sim B \sim C = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{matrix} x_1 + x_3 + x_5 & = 0 \\ x_2 - 2x_3 + 3x_5 & = 0 \\ x_4 - x_5 & = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = -x_3 - x_5 \\ x_2 = 2x_3 - 3x_5 \\ x_4 = x_5 \end{matrix}$

. $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{pmatrix} \Rightarrow X = x_3 \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{pmatrix} Basis\ for\ Null\,(A) = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{pmatrix}, \dim(Null\,(A)) = 2$

every system of linear equations has one of the following solutions:(i) There is no solution.
is a unique solution.(ii) There are more than one solution.

5.Methods of solving system of linear Equations

(5.1) (Direction method)

(5.2) (Method of inversion)

Consider the matrix equation when ,then the system has a unique solution. Pre multiplying by , we have .
Thus , has only one solution if and is given by .

(5.3) Example

Use the inverse of the matrix A in example (3.6) to solve the system:

Solution:

This system is equivalent to , , , ,So that:

. $\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}, A^{-1} = \begin{pmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

(5.4) Example

Solve the system: $\begin{matrix} 2x_1 + x_2 & = & 6 \\ 4x_1 + 2x_2 & = & 8 \end{matrix}$

Solution:

$A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, B = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$ But $\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0$, then has not inverse ,hens cannot solution by this method.

(5.5) Example

Solve the system: $\begin{matrix} -x_1 + x_2 + 2x_3 & = 1 \\ 3x_1 - x_2 + x_3 & = 1 \\ -x_1 + 3x_2 + 4x_3 & = 1 \end{matrix}$

Solution

$$A=\begin{pmatrix}-1 & 1 & 2\\ 3 & -1 & 1\\ -1 & 3 & 4\end{pmatrix},\ X=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix},\ B=\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$

$$A^{-1}=\frac{1}{10}\begin{pmatrix}-7 & 2 & 3\\ -13 & -2 & 7\\ 8 & 2 & -2\end{pmatrix}\Rightarrow\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\frac{1}{10}\begin{pmatrix}-7 & 2 & 3\\ -13 & -2 & 7\\ 8 & 2 & -2\end{pmatrix}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}=\begin{pmatrix}\frac{-1}{5}\\ \frac{-4}{5}\\ \frac{4}{5}\end{pmatrix}$$

(Using Elementary row operations:(Gaussian Elimination) We list the basic steps of Gaussian Elimination, a method to solve a system of linear equations. Except for certain special cases, Gaussian Elimination is still of the art."" After outlining the method, we will give some examples.

Gaussian elimination is summarized by the following three steps:

1) Write the system of equations in matrix form. Form the augmented matrix. You omit the symbols for the variables, the equal signs, and just write the coefficients and the unknowns in a matrix. You should consider the matrix as shorthand for the original set of equations.

2) Perform elementary row operations to get zeros below the diagonal.

3) An elementary row operation is one of the following:

a) multiply each element of the row by a non-zero constant. b) switch two rows.

c) add (or subtract) a non-zero constant times a row to another row.

4) Inspect the resulting matrix and re-interpret it as a system of equations.

d) If you get 0 = a non-zero quantity then there is no solution.

e) If you get less equations than unknowns after discarding equations of the form 0=0 and if there is a solution then there is an infinite number of solutions

f) If you get as many equations as unknowns after discarding equations of the form 0=0 and if there is a solution then there is exactly one solution.

Use Gaussian elimination to solve the system of linear equations:

$$\begin{aligned} x_1+5x_2 &= 7\\ -2x_1-7x_2 &= -5\end{aligned}$$

We carry out the elimination procedure on both the system of equations and the corresponding augmented matrix, simultaneously. In general, only one set of reductions is necessary, and the latter (dealing with matrices only) is preferable because of the simplified notation.

$$\begin{aligned} x_1+5x_2 &= 7\\ -2x_1-7x_2 &= -5\end{aligned}\Rightarrow\begin{pmatrix}1 & 5 & 7\\ -2 & -7 & -5\end{pmatrix}.$$

Add twice row 1 to row 2 $\Rightarrow \begin{pmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{pmatrix}$

Multiply row 2 by . $\frac{1}{3} \Rightarrow \begin{pmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{pmatrix}$

$$\Rightarrow \begin{matrix} x_1 = -8 \\ x_2 = \ \ 3 \end{matrix} \quad \begin{matrix} x_1 + 5x_2 = 7 \\ x_2 = 3 \end{matrix}$$

And we write the matrix by : $\begin{pmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{pmatrix}$

Use Gaussian elimination to solve the system of linear equations:

$$\begin{aligned} 2x_2 + x_3 &= -8 \\ x_1 - 2x_2 - 3x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 3 \end{aligned}$$

As before, we carry out reduction on the system of equations and on the augmented matrix simultaneously, in order to make it clear that row operations on equations correspond exactly to row operations on matrices.

$$\begin{aligned} 2x_2 + x_3 &= -8 \\ x_1 - 2x_2 - 3x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 3 \end{aligned} = \begin{pmatrix} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{pmatrix}$$

Row 1 and Row 2: $= \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ -1 & 1 & 1 & 2 \end{pmatrix}$

Row 1 to Row 3: $= \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{pmatrix}$

Swap Row 2 and Row 3: $= \begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & -8 \\ 0 & 2 & 1 & 3 \end{pmatrix}$

Add twice Row 2 to Row 3: $=\begin{pmatrix}1 & -2 & -3 & 0\\ 0 & -1 & -1 & 3\\ 0 & 0 & -1 & -2\end{pmatrix}$

$=\begin{pmatrix}1 & -2 & 0 & 6\\ 0 & -1 & 0 & 5\\ 0 & 0 & -1 & -2\end{pmatrix}=\begin{pmatrix}1 & 0 & 0 & -4\\ 0 & -1 & 0 & 5\\ 0 & 0 & -1 & -2\end{pmatrix}$ Add -2 times

Row 2 to Row 1:

Multiply Row 2 and Row 2 by -1: $=\begin{pmatrix}1 & 0 & 0 & -4\\ 0 & 1 & 0 & -5\\ 0 & 0 & 1 & 2\end{pmatrix}$

$\Rightarrow x_1=-4\ ,\ x_2=-5\ ,\ x_3=2$

.

(5.8) Example

$x+y+z=6$

. $2x-y+z=3$ First form the augmented matrix: $\begin{pmatrix}1 & 1 & 1 & 6\\ 2 & -1 & 1 & 3\\ 1 & 0 & 1 & 4\end{pmatrix}$

$x+z=4$

Next add -2 times the first row to the second row and then add -1 times the first row to the third row: $=\begin{pmatrix}1 & 1 & 1 & 6\\ 0 & -3 & -1 & -9\\ 1 & -1 & 1 & -2\end{pmatrix}$

Next multiply the second row by -1 and the third row by -1, just to get rid of the minus signs. Then switch the second and third rows: $=\begin{pmatrix}1 & 1 & 1 & 6\\ 0 & 1 & 0 & 2\\ 0 & 3 & 1 & 9\end{pmatrix}$

Now add -3 times the second row to the third row, so we have all zeros below the diagonal: $=\begin{pmatrix}1 & 1 & 1 & 6\\ 0 & 1 & 0 & 2\\ 0 & 3 & 1 & 3\end{pmatrix}$

.

re-interpret the augmented matrix as a system of equations, starting at the bottom and working backwards (this is called back substitution). The bottom equation is . $0x + y + 0z = 2 \Rightarrow y = 2$ next to the bottom equation is .The next equation (the top one) is . $x + y + z = 6$ Substitute the values and into the equation and get .

6. Matrix Factorization

A matrix decomposition or matrix factorization is a factorization of a matrix into a product of matrices in the mathematical discipline of linear algebra. There are numerous decomposition matrices to choose from; each find use among a particular class of problems. For instance, when solving system of linear equation , the matrix can be decomposed via decomposition (IS a lower triangular matrix and an upper triangular matrix), or decomposition (an orthogonal matrix (or unitary matrix) and am upper triangular matrix) or decomposition (an upper triangular matrix and a diagonal matrix).

(6.1) The Factorization

The LU factorization, as defined below, is motivated by the challenge of solving a series of equations with the same coefficient matrix $Ax = b_1, Ax = b_2, \ldots\ldots, Ax = b_p \quad (6)$

When is invertible, one could compute and then compute , and so on. However, it is more efficient to solve the first equation in sequence (6) by row reduction and obtain an LU factorization of at the same time. There

after, the remaining equations in sequence (6) are solved with the factorization. At first, assume that is an matrix that can be row reduced to echelon form, without row interchanges. (Later, we will treat the general case.) Then can be written in the form , where L is an lower triangular matrix with on the diagonal and is an echelon form of . For instance, Such a factorization is called an factorization of . The matrix L is known as a unit lower triangular matrix because it is invertible.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix} \begin{pmatrix} \square & * & * & * & * \\ 0 & \square & * & * & * \\ 0 & 0 & 0 & \square & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Before studying how to construct and , we should look at why they are so useful. When , the equation can be written as . Writing for , we can find by solving the pair of equations:

$$\begin{aligned} Ly &= b \\ Ux &= b \end{aligned}$$

First solve for ; and then solve for . Each equation is easy to solve because and are triangular.

Calculate the LU decomposition of the following matrix .. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{pmatrix}$

Solution:

1- By using the Gaussian Elimination to get the upper triangular matrix .

$$\begin{pmatrix}1 & 2 & 3\\ 2 & -4 & 6\\ 3 & -9 & -3\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 3\\ 0 & -8 & 0\\ 0 & -15 & -12\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & -15 & -12\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & -12\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

2-Form the lower triangular matrix by using the rules mentioned above for the row operations involved to get .

Start with the identity matrix: $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$.

Row operation 1: $R_2 - 2R_1 \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

Row operation 2: $R_3 \to R_3 - 3R_1 \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 2 & -4 & 6\\ 3 & -9 & -3\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 3\\ 0 & -8 & 0\\ 0 & -15 & -12\end{pmatrix}$

Row operation 3: $-\frac{1}{8}R_1 \begin{pmatrix}1 & 0 & 0\\ 2 & -8 & 0\\ 3 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 0 & -8 & 0\\ 0 & -15 & -12\end{pmatrix}, \sim \begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & -15 & -12\end{pmatrix}$

Row operation 4: . $R_3 + 15R_2 \begin{pmatrix}1 & 0 & 0\\ 2 & -8 & 0\\ 3 & -15 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & -15 & -12\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & -12\end{pmatrix}$

Row operation 5: . $-\frac{1}{6}R_3 \begin{pmatrix}1 & 0 & 0\\ 2 & -8 & 0\\ 3 & -15 & -12\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & -12\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

Thus decomposition is given by: . $\begin{pmatrix}1 & 2 & 3\\ 2 & -4 & 6\\ 3 & -9 & -3\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 2 & -8 & 0\\ 3 & -15 & -12\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

(6.3) Example

Solve the following system using an decomposition.

$x_1 + x_2 + x_3 = 5$

$2x_1 - 4x_2 + 6x_3 = 18$

$3x_1 - 9x_2 - 3x_3 = 6$

$Ax = b \begin{pmatrix}1 & 2 & 3\\ 2 & -4 & 6\\ 3 & -9 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}5\\ 18\\ 6\end{pmatrix}$ 2- Find an decomposition for. This will yield the equation .

Note:

We found the composition for earlier. It is given by:

$\begin{pmatrix}1 & 2 & 3\\ 2 & -4 & 6\\ 3 & -9 & -3\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 2 & -8 & 0\\ 3 & -15 & -12\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, so that: $\begin{pmatrix}1 & 0 & 0\\ 2 & -8 & 0\\ 3 & -15 & -12\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}5\\ 18\\ 6\end{pmatrix}$

And $y = U x \rightarrow \begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$.

3- let $y = U x$,then solve the equation $Ly = b$ for . y

. $\begin{pmatrix}1 & 0 & 0\\ 2 & -8 & 0\\ 3 & -15 & -12\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}5\\ 18\\ 6\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0 & 0\\ 2 & -8 & 0\\ 3 & -15 & -12\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}=\begin{pmatrix}5\\ 18\\ 6\end{pmatrix}$

$$y_1 = 5 , 2y_1 - 8y_2 = 18 \Rightarrow y_2 = -1 , 3y_1 - 15y_2 - 12y_3 = 6 \Rightarrow y_3 = 2$$

4-Take the values for and solve the equation for . This will give the solution to the system . $Ax = b$

. $\begin{pmatrix}-5\\ -1\\ 2\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}\Rightarrow x_3 = 2, x_2 = -1 , x_1 + 2x_2 + 3x_3 = 5 \Rightarrow x_1 = 1$

Therefore, the solution to the system : $x_1 = 1, x_2 = -1 , x_3 = 2$

7.Numerical Implementation

In this section we study algorithm of Gaussian elimination, and compare the solution of liner equation between hand calculus and by the algorithm for this decomposition.

(7.1) Matlab Function of Gaussian Elimination

function x = gauss (A, b)

```
[n,n] = size (A);
[n,k] = size (b);
x=zero(n,k);
for i=1: n-1
m=-A (i+1: n, i) / A (i, i);
A (i+1: n, :)=A (i+1: n, :)+m*A (i, :)
b (i+1: n :)+m*b (i, :);
end
end
```

let , we solve this system by above a logarithm , we obtain

(7.2) Example

```
>> A=[1,5;2,-7];
>> b=[7;-5];
>>x=gauss(A,b)
x=
    15.33
    -1.6667
```

But by hand calculus

8. Matlab Function of Factorization

(8.1) Numerical Notes:

Let be matrix . We assume is large , say .

1- Computing an factorization of takes about flops , whereas finding requires about flops .
2- Solving and requires about flops.
3- Multiplication of by also requires flops, but the result may not bae as accurate as that obtained from and .
4- If is sparse , then and may be sparse too, whereas is likely to be non spares .

```
function [L, U] = ludecomposition (A)
[m, n] =size(A);
if (n=m)
error ('the matrix must be square')
end
for j=1: n,
if (abs (j, j)) <1e-6)
error ('one of the pivot coefficient is zero – pivoting is necessary')
end
for k=j+1: n,
A (k, j) =A (k, j) / A (j, j);
A (k, j+1: n) =A (k, j+1: n)-A (k, j) *A (j, j+1: n); % Row operation
end
end
U=triu (A);
L=tril (A);
For j=1: n,
L (j, j) =1;
end
```

end

(8.2) Example

```
>> A= [1,2,3;2, -4,6;3, -9, -3];
>> [L, U] =ludecomposition(A)
Ans=
        1
Ans=
        1
Ans=
        1

L=
      1.000           0           0
      2.000      1.0000           0
      3.0000     1.8750       1.000
U=
    1    2     3
    0   -8     0
    0    0   -12
```

But by hand . $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -5 & -12 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(8.3) Code of Solution of the System by Factorization

Let , then .Let ,solve this equation , is upper triangle matrix.

$$L = \begin{pmatrix} L_{1\,1} & 0 & 0 & \dots\dots & 0 \\ L_{2\,1} & L_{2\,2} & 0 & \dots\dots & 0 \\ .. & \dots\dots & \dots\dots & & 0 \\ . & & & & \\ .L_{n\,1} & L_{n\,2} & L_{n\,3}. & \dots & L_{m\,m} \end{pmatrix} \quad , \; Ly = b \Rightarrow \begin{pmatrix} L_{1\,1} & 0 & 0 & \dots\dots & 0 \\ L_{2\,1} & L_{2\,2} & 0 & \dots\dots & 0 \\ .. & \dots\dots & \dots\dots & & 0 \\ . & & & & \\ .L_{n\,1} & L_{n\,2} & L_{n\,3}. & \dots & L_{m\,m} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ y_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{pmatrix}$$

Then , and , then .

Now we solve the equation , is lower triangle.

$$\begin{pmatrix} U_{11} & U_{12} & U_{13} & \cdots\cdots & U_{1m} \\ 0 & U_{22} & \cdots\cdots & & U_{2n} \\ .. & \cdots\cdots & \cdots\cdots & & \cdots \\ . & & & & \\ 0 & 0 & 0.\ \cdots & & U_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ y_m \end{pmatrix} \quad u_{mn} x_n = y_m \to x_n = \frac{y_m}{u_{mm}}$$ Then, .

$$x_i = y_i - \sum \frac{u_{ij} x_j}{u_{ii}}$$

```
function x=linsys-LU (A, b)
[m.n] = size(A);
if (m=n),
error ('A is not square ');
end
if (length(b)=n),
error ('b does not have the right lenght'),
end
[L, U] =ludecomposition (A);
y=zeros(n,1);
for k=1: n,
y(k)=b(k)-L (k,1: k-1) *y (1: k-1);
end
x=zeros (n,1);
for j=n: -1:1,
x (j) =(y(j)-U (j, j+1: n) *x (j+1: n)) /U (j, j);
end

end
```

(8.4) Example

```
>> A=[1,2,3;2,-4,6;3,-9,-3];
>> b=[5;18;6];
>> x=linsys-LU(A,b)
x=
    0.1818
   -1
   0.5
```

But by hand calculus we obtained $x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Conclusion

This paper Included the definition of a system of linear and nonlinear ways of solving linear and we talked about a special case of the system Ax=b in this case is b=0.

this case we called the system is homogeneous linear system.And we also talked about ways to solve the system Ax = b this method is direct including Method of inversion and Gaussian elimination and anther
indirect is the analysis of the matrix by LU factorization.

We found a slight difference in the method of solving with algorithm and manual solution.

Acknowledgment:

The authors would like to thanks the Deanship of Scientific Research at Prince Sattam Bin Abdulaziz University, Alkharj, Saudi Arabia for the assistance.

References:

[1] David C. Lay. Linear Algebra and Its Applications. Gilbert Strang. Four The Edition. University of Maryland—College.[3] Gregoirc Allaire and Sidi mohmoud kaber. Numerical linear algebra 4–internet.