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Solution Of Linear Systems By Using QR Factorization On Matlab

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Abstract

Includes paper on the definition of a system of linear and nonlinear way of solving linear and we talked about a special case of the system $Ax = b$ in this case is $b = 0$. And we also talked about way to solve the system $Ax = b$ (method of direct and indirect). And to talked about algorithm for QR way to solve the system and your comparing between method by hand.

Key words (linear system, matrices, concept of factorization, method solution of linear system)

1.Introduction

Linear algebra is based on systems of linear equations, and they introduce some of the most important concepts in a straightforward and tangible manner. In several sections, a systematic method for solving systems of linear equations is described.

Throughout the essay, this algorithm will be utilized for computations. Demonstrate how a vector equation and a matrix equation can be used to represent a system of linear equations.

As a result of this equivalence, problems requiring linear combinations of vectors will be reduced to queries regarding systems of linear equations. The fundamental concepts of spanning and linear independence, as we study the beauty and power of linear algebra, it will play an important role throughout the text. The importance of linear algebra in applications has risen in lockstep with the advancement of computing power, with each new generation of hardware and software triggering a demand for even greater capabilities. Through the rapid rise of parallel processing and large-scale computations, computer science has become inextricably intertwined with linear algebra. Scientists and technologists are today working on issues that were unimaginable only a few decades ago. Today, linear algebra has more potential utility for students than any other collegiate mathematics course in many scientific and business sectors. The information in this publication lays the groundwork for future research in a variety of fascinating fields. Here are a few alternatives; we'll go through more later. Exploration for oil.

While searching for offshore oil resources, a ship's computers calculate thousands of independent systems of linear equations per day. Waves bounce off subterranean rocks are measured by geophones attached to mile-long cables behind the ship.

2. Linear Equation Systems (Linear System)

In mathematic we define a linear equation by the form:

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots\ldots\ldots\ldots + a_nx_n = b \qquad (1)$$

Where $a_1,\ldots,a_n$ and b is real or complex numbers, for example:

$$2x_1 + x_2 - x_3 = 2 \qquad (2)$$

$$4x_1 - 5x_2 = x_1x_2 \qquad (3)$$

$$x_2 = 2\sqrt{x_1} - 5 \qquad (4)$$

Equation (3) and (4) are not liner because of the presence x_1x_2 in the second and

$\sqrt{x_1}$ in the there'd equation.

A set of one or more linear equations containing a number of variables.

$x_1,\ldots\ldots x_n$

Said to be system of linear equations for example:

$$2x_1 - x_2 + 5x_3 = 8$$

$$x_1 - 4x_3 = -7$$

(2.1) The matrix equation $AX = B$

If $A_{n\times m}$ is matrix has n rows and m columns and $a_1,\ldots,a_n \in R$, $x \in R^n$, then the product of A and X denoted by AX , is the linear combination of the columns of A using the corresponding entries in X as weights that is :

$$AX = (a_1 \ a_2 \ldots a_n)\begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} = a_1x_1 + a_2x_2 + a_3x_3 + \ldots\ldots\ldots\ldots + a_nx_n \qquad (5)$$

Since the number of columns in A must equal the number of entries in X , AX is defined.

(2.2) Homogeneous Linear Systems

A system of linear equations is said to be homogeneous if it can be written in the form $AX = 0$, where A is an $n \times m$ matrix and 0 is the zero vector in R^n.
Such a system $AX = 0$ always has at least one solution, namely ,x = 0 (the zero vector in R^n . The trivial solution is the name given to this zero solution. For
a given equation $AX = 0$, the important question is whether there exists a nontrivial solution, that is, a nonzero vector x that satisfies $AX = 0$.
The homogeneous equation $AX = 0$ has a nontrivial solution if and only if the equation has at least one free variable.

3. Solution of Linear System

(3.1) Solving the System $AX = 0$

We define the method of solution the system by next example**:**

(3.2) Example

Determine if there is a nontrivial solution to the following homogeneous system. Then describe the solution set.

$3x_1 + 5x_2 - 4x_3 = 0$
$-3x_1 - 2x_2 + 4x_3 = 0$
$6x_1 + x_2 - 8x_3 = 0$

Solution:

Let A be the matrix of coefficients of the system and row reduce the augmented matrix $[A = 0]$ to echelon form:

$$\begin{pmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$AX = 0$ has nontrivial solutions (one for each choice of x_3) since x_3 is a free variable. To describe the solution set, continue the row reduction of $[A_0]$ to reduced echelon form:

$$\begin{pmatrix} 1 & 0 & \frac{-4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$, since

$x_1 - \frac{4}{3}x_3 = 0$
$x_2 = 0$
$0 = 0$

Solve for the basic variables x_1 and x_2 , and obtain $3x_1 = 4x_3,\ x_2 = 0$,with x_3 free. As a vector, the general solution of Ax = 0has the form:

$$X = \begin{pmatrix} \frac{3}{4} \\ 0 \\ 1 \end{pmatrix}$$

(3.3) Solving the system $AX = B$

In linear algebra, solution sets of linear systems are fundamental objects to study. They will appear later in several different contexts.

(3.4) The inverse of a matrix

A matrix $A_{n\times m}$ is said to be invertible if there is an $n \times n$ matrix C such that $CA = I$ and $AC = I$ where $I = I\,n$; then $n \times n$ is identity matrix. In this case, C is an inverse of A . In fact, C is uniquely deter If mined by A ,because if B were another inverse of A , then $B = BI = B(AC) = (BA)C = IC = C$ This unique inverse is denoted by A^{-1}, so that $A^{-1}A = AA^{-1} = I$. A Singular matrix are invertible matrices that are not invertible sometimes., and an
invertible matrix is called a nonsingular matrix.

(3.5) Determinate of matrices

Let $A = (a_{i,j})_{n\times n}$ be square matrix of order n , then the number $|A|$ is called determinate of the matrix A .

(i) Determinate of 2×2 matrix , the matrix A has inverse if the determinant is not zero. let $ad - bc$, if $ad - bc = 0$,then A is not invertible and $ad - bc$ is called determinate of A , and denoted by $\det(A)$.

(ii) Determinate of 3×3 matrix .Let $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then

$$|B| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - gf) + c(dh - ge).$$

(3.6) Note

If $|B| = 0$, then B has not inverse and called singular matrix.

(3.7) Example

Find the inverse of $A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$

Solution

Since $\det = (3.6) - (4.5) = -2 \neq 0$, A is invertible, and $A^{-1} = \frac{-1}{2}\begin{pmatrix} 6 & -5 \\ -4 & 3 \end{pmatrix}$ Invertible matrices are indispensable in linear algebra mainly for algebraic calculations and formula derivations, as in the next theorem. There are also occasions when an inverse matrix provides insight into a mathematical model of a real-life situation, as in Example, below.

(3.8) Theorem (Unique Solution)

If A is an invertible $n \times n$ matrix, then for each $B \in R^n$, the equation $AX = B$ has the unique solution is $X = A^{-1}B$.

Proof:

Take any $B \in R^n$. A solution exists because if $A^{-1}B$ is substituted for X, then $AX = A(A^{-1}B) = (A\,A^{-1})B = I\,B = B$ .So $(A^{-1}I\,B)$ is a solution. To prove that the solution is unique, show that if U is any solution, then U ; in fact, must be $A^{-1}B$ indeed, if $AU = B$, we can multiply both sides by A^{-1} and obtain; $A^{-1}A\,U = A^{-1}B$,$U = A^{-1}B$.

4.Types of matrices

(4.1) (Square Matrix)

A square matrix has the same number of rows and columns as the number of columns.

(4.2) (Diagonal Matrix)

A square matrix $A = (a_{i,j})_{n*n}$ called a diagonal matrix if each of its non-diagonal element is zero. That is $a_{ij} = 0$ if $i \neq j$, and at least one element $a_{ij} \neq 0$.

(4.3) (Identity Matrix)

A diagonal matrix whose diagonal elements are equal to 1 is called identity matrix and denoted by I_n . That is $\begin{pmatrix} 0, i \neq j \\ 1, i = j \end{pmatrix}$.

(4.4) (Upper Triangular Matrix)

A square matrix said to be a upper triangular matrix if $a_{ij} = 0, i > j$.

(4.5) (Lower Triangular Matrix)

A square matrix said to be a Lower triangular matrix if $a_{ij} = 0, i < j$.

(4.6) Note:

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A .

(4.7) (Symmetric Matrix)

A square matrix $A = (a_{ij})_{n\times n}$ is said to be a symmetric if $a_{ij} = a_{ji}, \forall i, j$.

(4.8) (Column Vector)

A column vector or column matrix is a matrix with only one column.

(4.9) (Elementary Matrix)

An elementary matrix is one that is produced from an identity matrix by conducting a single elementary row operation. The three types of elementary matrices are demonstrated in the following example.

(4.10) Example

Let: $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}, A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, calculate

$E_1 A, E_2 A, E_3 A$, and describe how these products can be obtained by elementary row operation on A .

Solution

$$E_1 A = \begin{pmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{pmatrix}, E_2 A = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}, E_3 A = \begin{pmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{pmatrix}.$$

Addition of −4 times row 1 of A to row 3 produces $E_1 A$, (This is a row re- placement operation.) An interchange of rows 1 and 2 of A produces $E_2 A$, and multiplication of row 3 of A by 5 produces $E_3 A$.

(4.11) Definition

An indexed set of vectors $v_1, \ldots\ldots, v_p$ is said to be linearly independent if the vector equation $c_1 v_1 + \ldots\ldots + c_p v_p = 0$ has only the trivial solution $c_1 = c_2 = \ldots.. = c_p = 0$ .If there is $c_i \neq 0$,Then the set is said to be linear dependent.

(4.12) (Column Space)

A set of all linear combination of the column of A , is called column space.

(4.13) (Rank of Matrix)

The rank of a matrix A , denoted by rank A , is the dimension of the column space of A .

(4.14) (Null Space of Matrix)

The null space of a matrix A is the set $Nul\ (A)$ of all solutions of the homogeneous equation $AX = 0$.

(4.15) (Basis of Matrix)

A basis for a subspace A of R^n is a linearly independent set in A that spans A .

(4.16) (The Pivot Column of Matrix)

The pivot columns of a matrix A form a basis for the column space of A .

(4.17) (Dimensional of Matrix)

If V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V , written as $\dim(V)$, is the number of vectors in a basis for V . The dimension of the zero vector space0 is defined to be zero. If V is not spanned by a finite set, then V is said to be infinite-dimensional

(4.18) (The Row Space of Matrix)

The set of all linear combination of the rows of A ,and denoted by $row\ (A)$.

(4.19) Theorem

If two matrices A and B are row equivalent, then their row space the same.

If B in echelon form the non-zero rows of B form a basis for the row space of A as well as the form B .

(4.20) Example

Find bases for the row space, the null space of the matrix:

$$A = \begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix} \quad A \sim B = \begin{pmatrix} 1 & 3 & -5 & 0 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution:

$Rwo\ (A) = [(1,3,-5,1,5),(0,1,-2,2,-7),(0,0,0,-4,20)]$.

$Col\,(A)=\left\{\begin{pmatrix}-2\\1\\3\\1\end{pmatrix},\begin{pmatrix}-5\\3\\11\\7\end{pmatrix},\begin{pmatrix}0\\1\\7\\5\end{pmatrix}\right\}$, to find $Null\,(A)$,we have to find the reduced echelon form of A .

$A\sim B\sim C=\begin{pmatrix}1&0&1&0&1\\0&1&-2&0&3\\0&0&0&1&-5\\0&0&0&0&0\end{pmatrix}$, and $\begin{matrix}x_1+x_3+x_5=0\\x_2-2x_3+3x_5=0\\x_4-x_5=0\end{matrix}\Rightarrow\begin{matrix}x_1=-x_3-x_5\\x_2=2x_3-3x_5\\x_4=x_5\end{matrix}$

$$X=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}-x_3-x_5\\2x_3-3x_5\\x_3\\5x_5\\x_5\end{pmatrix}\Rightarrow X=x_3\begin{pmatrix}-1\\2\\1\\0\\0\end{pmatrix}+x_5\begin{pmatrix}-1\\-3\\0\\5\\1\end{pmatrix}Basis\ for\ Null\,(A)=\begin{pmatrix}-1\\2\\1\\0\\0\end{pmatrix},\begin{pmatrix}-1\\-3\\0\\5\\1\end{pmatrix},\dim(Null\,(A))=2\,.$$

As a result, every system of linear equations has one of the following solutions:

(i) There is no solution.

(ii) There is a unique solution.

(ii) There are more than one solution.

5.Methods of solving system of linear Equations

(5.1) (Direction method)

(5.2) (Method of inversion)

Consider the matrix equation $AX=B$ when $|A|\neq 0$,then the system has a unique solution. Pre multiplying by A^{-1}, we have $A^{-1}(AX)=A^{-1}B\ X\Rightarrow X=A^{-1}B$.

Thus $AX=B$, has only one solution if $|A|\neq 0$ and is given by $X=A^{-1}B$.

(5.3) Example

Use the inverse of the matrix A in example (3.6) to solve the system:

$3x_1+4x_2=3$

$5x_1+6x_2=7$

Solution:

This system is equivalent to $AX=B$, $A=\begin{pmatrix}3&4\\5&6\end{pmatrix}$, $X=\begin{pmatrix}x_1\\x_2\end{pmatrix}$, $B=\begin{pmatrix}3\\7\end{pmatrix}$,So that:

$$\begin{pmatrix}3&4\\5&6\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}3\\7\end{pmatrix},\ A^{-1}=\begin{pmatrix}-3&2\\\frac{5}{2}&-\frac{3}{2}\end{pmatrix}\Rightarrow\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}-3&2\\\frac{5}{2}&-\frac{3}{2}\end{pmatrix}\begin{pmatrix}3\\7\end{pmatrix}=\begin{pmatrix}5\\-3\end{pmatrix}.$$

(5.4) Example

Solve the system: $\begin{matrix} 2x_1 + x_2 = 6 \\ 4x_1 + 2x_2 = 8 \end{matrix}$

Solution:

$A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, B = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$, But $\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0$, then has not inverse ,hens cannot solution by this method.

(5.5) Example

Solve the system: $\begin{matrix} -x_1 + x_2 + 2x_3 = 1 \\ 3x_1 - x_2 + x_3 = 1 \\ -x_1 + 3x_2 + 4x_3 = 1 \end{matrix}$

Solution

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{10}\begin{pmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-1}{5} \\ \frac{-4}{5} \\ \frac{4}{5} \end{pmatrix}.$$

(5.6) (Using Elementary row operations:(Gaussian Elimination)

We list the basic steps of Gaussian Elimination, a method to solve a system of linear equations. Except for certain special cases, Gaussian Elimination is still of the art."" After outlining the method, we will give some examples.

Gaussian elimination is summarized by the following three steps:

1) Write the system of equations in matrix form. Form the augmented matrix. You omit the symbols for the variables, the equal signs, and just write the coefficients and the unknowns in a matrix. You should consider the matrix as shorthand for the original set of equations.

2) Perform elementary row operations to get zeros below the diagonal.

3) An elementary row operation is one of the following:

a) multiply each element of the row by a non-zero constant. b) switch two rows.

c) add (or subtract) a non-zero constant times a row to another row.

4) Inspect the resulting matrix and re-interpret it as a system of equations.

d) If you get 0 = a non-zero quantity then there is no solution.

e) If you get less equations than unknowns after discarding equations of the form 0=0 and if there is a solution then there is an infinite number of solutions

f) If you get as many equations as unknowns after discarding equations of the form 0=0 and if there is a solution then there is exactly one solution.

(5.6) Example

Use Gaussian elimination to solve the system of linear equations:

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned}$$

Solution:

We carry out the elimination procedure on both the system of equations and the corresponding augmented matrix, simultaneously. In general, only one set of reductions is necessary, and the latter (dealing with matrices only) is preferable because of the simplified notation.

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned} \Rightarrow \begin{pmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{pmatrix}.$$

Add twice row 1 to row 2 $\Rightarrow \begin{pmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{pmatrix}$

Multiply row 2 by $\frac{1}{3} \Rightarrow \begin{pmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{pmatrix}.$

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ x_2 &= 3 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -8 \\ x_2 &= 3 \end{aligned}$$

And we write the matrix by : $\begin{pmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{pmatrix}$

(5.7) Example

Use Gaussian elimination to solve the system of linear equations:

$$\begin{aligned} 2x_2 + x_3 &= -8 \\ x_1 - 2x_2 - 3x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 3 \end{aligned}$$

Solution:

As before, we carry out reduction on the system of equations and on the augmented matrix simultaneously, in order to make it clear that row operations on equations correspond exactly to row operations on matrices.

$$\begin{aligned} 2x_2 + x_3 &= -8 \\ x_1 - 2x_2 - 3x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 3 \end{aligned} = \begin{pmatrix} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{pmatrix}$$

Swap Row 1 and Row 2: $=\begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ -1 & 1 & 1 & 2 \end{pmatrix}$

Add Row 1 to Row 3: $=\begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{pmatrix}$

Swap Row 2 and Row 3: $=\begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & -8 \\ 0 & 2 & 1 & 3 \end{pmatrix}$

Add twice Row 2 to Row 3: $=\begin{pmatrix} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & -1 & -2 \end{pmatrix}$

Add -1 times Row 3 to Row 2, add -3 times Row 3 to Row 1: $=\begin{pmatrix} 1 & -2 & 0 & 6 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & -1 & -2 \end{pmatrix}$

Add -2 times Row 2 to Row 1: $=\begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & -1 & -2 \end{pmatrix}$

Multiply Row 2 and Row 2 by -1: $=\begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

$\Rightarrow x_1 = -4 \ , x_2 = -5 \ , x_3 = 2$.

(5.8) Example

$x + y + z = 6$

$2x - y + z = 3$.

$x + z = 4$

First form the augmented matrix: $\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 0 & 1 & 4 \end{pmatrix}$

Next add -2 times the first row to the second row and then add -1 times the first row to the third row:

$$=\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 1 & -1 & 1 & -2 \end{pmatrix}$$

Next multiply the second row by -1 and the third row by -1, just to get rid of the minus signs. Then switch the second and third rows:

$$=\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 9 \end{pmatrix}$$

Now add -3 times the second row to the third row, so we have all zeros below the diagonal:

$$=\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 3 \end{pmatrix}.$$

Now re-interpret the augmented matrix as a system of equations, starting at the bottom and working backwards (this is called back substitution). The bottom equation is $0x + 0y + z = 3 \Rightarrow z = 3$. The next to the bottom equation is $0x + y + 0z = 2 \Rightarrow y = 2$.

The next equation (the top one) is $x + y + z = 6$ .Substitute the values $z = 3$ and $y = 2$ into the equation and get $x = 1$.

6.The QR Factorization

Let A be a matrix we called an orthogonal matrix if their column is orthonormal.

(6.1) An orthonormal

i) orthogonal with any either column.

ii) norm of any column is equal one.

Note:

We use this factorization if column of matrix is linearly independent.

The main idea of the QR factorization is again to reduce a linear system to a triangular one. However, the matrix is not factorized as the product of two triangular matrices (as previously), but as the product of an upper triangular matrix R and an orthogonal (unitary) matrix Q , which, by definition, is easy to invert, since $Q^{-1} = Q^*$.

In order to solve the linear system $Ax = b$ we proceed in three steps.

i) Factorization: finding an orthogonal matrix Q such that $Q^* A = R$ is upper triangular.

ii) Updating the right-hand side: computing $Q^* b$.

iii) Back substitution: solving the triangular system $R x = Q^* b$.

If A is non-singular, the existence of such an orthogonal matrix Q is guaranteed by the following result, for which we give a constructive proof by the Gram- Schmidt orthonormalization process.

(6.2) QR factorization via Gram-Schmidt

we start by formally writing down the QR factorization $A = QR$ as:

$$a_1 = q_1 r_{11} \Rightarrow q_1 = \frac{a_1}{r_{11}} \quad (6)$$

$$a_2 = q_1 r_{12} + q_2 r_{22} \Rightarrow q_2 = \frac{a_2 - q_1 r_{12}}{r_{22}} \quad (7)$$

$$a_n = q_n r_{1n} + q_2 r_{2n} + \dots\dots\dots + q_n r_n \Rightarrow q_n = \frac{a_n - \sum_{i n} q_i}{r_{n n}} \quad (8)$$

Note that is these formula the column a_j of A are given and we want to determine the column q_j of Q and entries $r_{i j}$ such that Q is orthonormal.

$$q_i^* q_j = s_{i j} \quad (9)$$

is upper triangular and $A = QR$. the latter two condition are already R reflected in the formulas above.

Using (6) in the orthogonality condition (9) we get:

$$q_1^* q_1 = \frac{a_1^* a_1}{r_{11}^2} = 1 \quad (10)$$

So that:

$$r_{11} = \sqrt{a_1^* a_1} = \|a_1\|_2 \quad (11)$$

Note that we arbitrarily chose the positive root here (so that the factorization becomes unique). Next, the orthogonality condition (9) give us:

$$q_1^* q_2 = 0 \text{ , } q_2^* q_2 = 1 \quad (12)$$

Now we apply (7) to the first of these two condition. Then:

$$q_1^* q_2 = \frac{q_1^* a_2 - r_{12} q_1^* q_1}{r_{2 2}} = 0 \quad (13)$$

Since we ensured $q_1^* q_1 = 1$ in the previous step the number yields $r_{1 2} = q_1^* a_2$ so that:

$$q_2 = \frac{a_2 - (q_1^* a_2)\, q_1}{r_{2 2}} \quad (14)$$

To find $r_{2\,2}$ we normalize demand that $q_2^* q_2 = 1$ or $\|q_2\|_2 = 1$ equivalently .This immediately gives:

$$r_{2\,2} = \| a_2 - (q_1^* a_2) q_1 \|_2 \quad (15)$$

To fully understand how the algorithm proceeds we add one more step (for $n = 3$).
Now we have three orthogonality condition:

$$q_1^* q_3 = q_2^* q_3 = q_3^* q_3 = 0 \quad (16)$$

The first of these condition together with (8) for ($n = 3$) yields:

$$q_1^* q_3 = \frac{q_1^* a_3 - r_{1\,3} q_2^* q_3 - r_{2\,3} q_2^* q_2}{r_{3\,3}} = 0 \quad (17)$$

so that $r_{1\,3} = q_1^* a_3$ due to the orthogonality of columns q_1 and q_2. Similarly, the second orthogonality condition together with (8) for ($n = 3$) yields:

$$q_2^* q_3 = \frac{q_2^* a_3 - r_{1\,3} q_2^* q_1 - r_{2\,3} q_2^* q_2}{r_{3\,3}} = 0 \quad (18)$$

So that: $r_{3\,3} = q_2^* a_3$

Together this give us:

$$q_3 = \frac{a_3 - (q_1^* a_3)\, q_1 - (q_2^* a_3)\, q_2}{r_{3\,3}} \quad (19)$$

and the last unknown $r_{3\,3}$, is determined by normalization

$$r_{3\,3} = \| a_3 - (q_1^* a_3)\, q_1 - (q_2^* a_3)\, q_2 \|_2 \quad (20)$$

(6.3) Theorem

Let $A \in C^m \times n$ with $m \geq n$. Then A has QR a factorization . Moreover , if A is of full rank(n) , then the reduced factorization $A = QR$ with $r_{j\,j} > 0$ is unique .

(6.4) Example

We compute the QR factorization for the matrix :

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

First $v_1 = a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$r_{11} = \|v_1\| = \sqrt{2}$,this us next $q_1 = \frac{v_1}{\|v_1\|}$ and $v_2 = a_2 - q_1^* a_2 q_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \frac{\sqrt{2}}{\sqrt{2}} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ .Thus calculation required that $r_{12} = \frac{2}{\sqrt{2}} = \sqrt{2}$, and $q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

In the third iteration we have $v_3 = a_3 - (q_1^* a_3) q_1 - (q_2^* a_3) q_2$, from which we first compute $r_{11} = \frac{1}{\sqrt{2}}$ and $r_{33} = 0$ .This gives us $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 0 = \frac{1}{2} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ .Finally, and $r_{33} = \|v_3\| = \frac{\sqrt{6}}{2}$ .Collecting all of the information we end up with $q_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{pmatrix}.$$

7.An Application of the QR Factorization

Consider solution of the liner system $Ax = b$ with $A \in C^{m \times m}$ nonsingular . Since $Ax = b \Leftrightarrow QRx = b \Leftrightarrow Rx = Q^{\bullet} b$,where the last equation holds since Q is unitary , we can proceed as follows:

1-Compute $A = QR$ (which is the same as $A = Q^{/} R^{/}$ in this case).

2- Compute $y = Q^{\bullet} b$.

3- Solve the upper triangular $Rx = y$.

We will have more application for the QR factorization in the context of least squares problems.

(7.1) Example

Using the QR factorization to solve the following system $Ax = b$:

$$\begin{aligned} x_1 - x_2 + 2x_3 &= -3 \\ -x_1 + x_2 \quad &= 1 \\ -2x_2 + x_3 &= 5 \end{aligned}$$

Solution

$$A=\begin{pmatrix}1 & -1 & 2\\ -1 & 1 & 0\\ 0 & -2 & 1\end{pmatrix}, b=\begin{pmatrix}-3\\ 1\\ 5\end{pmatrix}$$, then $$Q=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & -1 & 0\end{pmatrix}$$, $$R=\begin{pmatrix}\sqrt{2} & -\sqrt{2} & \sqrt{2}\\ 0 & 2 & -1\\ 0 & 0 & \sqrt{2}\end{pmatrix}$$, Since

$Ax=b\Rightarrow Rx=Q^T$,let $Q^Tb=y$ then $$Q^Tb=y=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ 0 & 0 & -1\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{pmatrix}\begin{pmatrix}-3\\ 1\\ 5\end{pmatrix}$$, to solve

$Rx=y$

$$\begin{pmatrix}\sqrt{2} & -\sqrt{2} & \sqrt{2}\\ 0 & 2 & -1\\ 0 & 0 & \sqrt{2}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}-4\\ \frac{5}{\sqrt{2}}\\ -2\end{pmatrix}\Rightarrow\sqrt{2}x_3=\frac{-2}{\sqrt{2}}\Rightarrow 2x_3=-1$$

$$2x_2-x_3=\left(-\frac{5}{\sqrt{2}}\right)(\frac{1}{2})\Rightarrow x_2=-3,\sqrt{2}x_1-\sqrt{2}x_2+\sqrt{2}x_3=-4\Rightarrow x_1=-4$$

8.Numerical Implementation

In this section we study algorithm of Gaussian elimination, and compare the solution of liner equation $Ax=b$ between hand calculus and by the algorithm for this decomposition.

(8.1) Matlab Function of Gaussian Elimination

```
function x = gauss (A, b)
[n,n] = size (A);
[n,k] = size (b);
x=zero(n,k);
for i=1: n-1
m=-A (i+1: n, i) / A (i, i);
A (i+1: n, :)=A (i+1: n, :)+m*A (i, :)
b (i+1: n :)+m*b (i, :);
end
end
```

let $A=\begin{pmatrix}1 & 5\\ -2 & -7\end{pmatrix}$, $b=\begin{pmatrix}7\\ -5\end{pmatrix}$, we solve this system $Ax=b$ by above a logarithm , we obtained :

(8.2) Example

```
>> A= [1,5;2,-7];
>> b= [7;-5];
>>x=gauss(A,b)
x=
     15.33
    -1.6667
```

But by hand calculus $x = \begin{pmatrix} -8 \\ 3 \end{pmatrix}$

9. Matlab function of the QR Factorization

(9.1) Code of QR :

```
function x= [Q, R] = myqr (A)
[m, n] = size (A);
R = zeros (n);
Q=A;
for k=1: n
   for i = 1: k-1
R (i, k) = Q (k) – R (i, k) * Q (k);
end
R (k, k) = norm (Q (: k));
Q (: k) =Q (: k) * R (k, k);
end
end
```

If $Ax = b$. Let $A = QR$,then $Ax = b \Rightarrow QRx = b$, Q orthogonal matrix ($Q_T = Q_{-1}$), $Rx = Q_T\, b$,let $C = Q_T\, b$, R is upper triangle matrix

$$Rx = c = \begin{pmatrix} r_{11} & r_{12} & .... & r_{1n} \\ 0 & r_{22} & ..... & r_{2n} \\ . & . & ..... & . \\ 0 & 0 & 0 & r_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ . \\ c_n \end{pmatrix}, \; x_i = \frac{c_i - \sum r_{ij}\, x_j}{r_{ii}}.$$

(9.2) Code of Solution of System $Ax = b$ by the QR Factorization

```
function x= solqr (A, b)
[Q, R] myqr (A);
C = transpose (Q) *b;
[m, n] = size (A);
X (n) = c (n) / R (n, n);
for i = n-1: -1: 1
      sum = 0;
      for j=i+1: n
           sum = sum + (R (i, j) * x (j));
      end
      x (i) = (c (i) – sum / R (i, i));
```

end

end

(9.3) Example

Use above logarithm to obtain Q and R , let $A=\begin{pmatrix}1 & -1 & 2\\ -1 & 1 & 0\\ 0 & -2 & 1\end{pmatrix}, b=\begin{pmatrix}-3\\ 1\\ 5\end{pmatrix}$ and solve $Ax=b$.

Solution

```
>> A= [1, -1,2; -1,1,0;0, -2,1];
>> b = [-3;1;5];
>> [Q, R] = myqr (A)
Q =

   1.4142   -2.4495    4.4721
  -1.4142    2.4495         0
        0         0    2.2361

R =

   1.4142   -1.4142         0
        0    2.4495         0
        0         0    2.2361


>> x = solqr (A,b)
x =
   -0.2400    -0.1667   -1.0000
```

And by calculus hand:

$$Q=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & -1 & 0\end{pmatrix}, R=\begin{pmatrix}\sqrt{2} & -\sqrt{2} & \sqrt{2}\\ 0 & 2 & -1\\ 0 & 0 & \sqrt{2}\end{pmatrix}, x=\begin{pmatrix}-4\\ -3\\ -1\end{pmatrix}$$

Conclusion

Includes research on the definition of a system of linear and nonlinear ways of solving linear and we talked about a special case of the system $Ax=b$ in this case is b=0.in this case we called the system is homogeneous linear system.

And we also talked about ways to solve the system $Ax = b$ this method is direct including Method of inverses ion and Gaussian elimination and anther in- direct is the analysis of the matrix with many ways like QR factorization.

And to talked about algorithm for some ways to solve the system and your comparing between the solution of this system through algorithms and calculation method by hand and we find a few difference between him.

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