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$(R^{*}G^{*})^{**}$-Closed Sets In Topological Spaces

N. Meenakumari & T.Indira

Associate professor, PG & Research Department of Mathematics, Seethalakshmi Ramaswami College, Trichy.

Abstract: The aim of this paper is to introduce a new class of closed sets namely (r*g*)**-closed sets which is obtained by generalizing (r*g*)*-closed sets via (r*g*)*open setsand investigate some of their basic properties in topological spaces

Keywords: (r*g*)*-closed sets, (r*g*)*-open sets.

1. Introduction

Levine introduced the class of g-closed sets. Many topologists have introduced several class of new sets and their properties. The authors [4] have already introduced (r*g*)*-closed sets and investigated some of their properties. The aim of this paper is to introduce $(r^{*}g^{*})^{**}$-closed sets by generalizing closed sets via $(r^{*}g^{*})^{*}$-open sets and investigate some properties.

2. Preliminaries

Definition 2.1: A subset A of a space X is called

1. A α-generalized closed (α g- closed) [5] set if α cl(A) $\subseteq$ U whenever A$\subseteq$ U and U is open.
2. A generalized semi closed (briefly gs - closed) [6] if scl(A)$\subseteq$ U
   whenever A$\subseteq$ U and U is open in X.
3. A g* closed [2] if cl(A) $\subseteq$ U whenever A$\subseteq$ U and U is g-open.
4. A g**closed[4] if cl(A) $\subseteq$ U whenever A$\subseteq$ U and U is g*-open.
5. A generalized pre -closed (gp)- closed [1] if pcl(A) $\subseteq$ U whenever
   A$\subseteq$ U and U is open.
6. A r*g*closed set [3] if rcl(A) $\subseteq$ U whenever A$\subseteq$ U and U is g- open.

Definition 2.2: A subset A of a topological space (X, τ) is called a (r*g*)*-closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is (r*g*)-open.

3. (r*g*)-closed sets.**

Definition 3.1: A subset Aof a topological spaces of (X, τ) is called a $(r^*g^*)^{**}$-closed set if $cl(A) \subseteq A$ whenever $A \subseteq U$ and U is $(r^*g^*)^*$-open. The compliment of $(r^*g^*)^{**}$-closed is called as a $(r^*g^*)^{**}$-open.

Proposition 3.2: Every closed set is $(r^*g^*)^{**}$-closed set.

Proof: Let $A \subseteq U$, where U is $(r^*g^*)^*$-open.
Since A is closed, $cl(A) = A$. Therefore $cl(A) = A \subseteq U$.
$\Rightarrow cl(A) \subseteq U \Rightarrow A$ is$(r^*g^*)^{**}$-closed set.
The converse need not be true as seen from the following example.

Example 3.3: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{a, c\}\}$
Closed sets are $\emptyset, X, \{a, b\}, \{b\}$.
$(r^*g^*)^{**}$-closed sets are $\emptyset, X, \{b\}, \{a, b\}, \{b, c\}$.
Here $\{b, c\}$ is $(r^*g^*)^{**}$-closed but not closed.

Proposition 3.4: Every $(r^*g^*)^{**}$-closed set is gs-closed set.

Proof: Let A be $(r^*g^*)^{**}$-closed. Let $A \subseteq U$, where U is open. Every open set is $(r^*g^*)^*$-open. Therefore $A \subseteq U \Rightarrow cl(A) \subseteq U$. But $scl(A) \subseteq cl(A) \subseteq U \Rightarrow scl(A) \subseteq U$ whenever $A \subseteq U$, U is open. Hence A is gs-closed set.
The converse need not be true as seen from the following example.

Example 3.5: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}\}$
$(r^*g^*)^*$-open sets are $\emptyset, X, \{a, c\}, \{c\}, \{a\}$.
$(r^*g^*)^{**}$-closed sets are $\emptyset, X, \{b\}, \{a, b\}, \{b, c\}$.
gs-closed sets are $\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}$.
Here $\{c\}$ is gs-closed but not $(r^*g^*)^{**}$-closed.

Proposition 3.6: Every $(r^*g^*)^{**}$-closed set is αg-closed set.

Proof: Let A be $(r^*g^*)^{**}$-closed. Let $A \subseteq U$,where U is open.But every open set is $(r^*g^*)^*$-open. We have $cl(A) \subseteq U$. But $\alpha cl(A) \subseteq cl(A) \subseteq U \Rightarrow \alpha cl(A) \subseteq U. \Rightarrow \alpha cl(A) \subseteq U \Rightarrow A$ is αg-closed set.
The converse need not be true as seen from the following example.

Example 3.7: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$
Closed sets are $\emptyset, X, \{b, c\}$
$(r^*g^*)^*$-closed sets are $\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}$
$(r^*g^*)^*$-open sets are $\emptyset, X, \{a, c\}, \{a, b\}, \{c\}, \{a\}, \{b\}$.
$(r^*g^*)^{**}$-closed sets are $\emptyset, X, \{b, c\}$
α-closed sets are $\emptyset, X, \{b\}, \{c\}, \{b, c\}$.
αg-closed sets are $\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}$.
Here $\{b\}, \{c\}, \{a, b\}, \{a, c\}$ is αg-closed but not $(r^*g^*)^{**}$-closed.

Proposition 3.8: Every $(r^*g^*)^{**}$-closed set is gp-closed set.

Proof: Proof follows from the fact that every open set is $(r^*g^*)^*$-open and $pcl(A) \subseteq cl(A) \subseteq U$.
The converse need not be true as seen from the following example.

Example 3.9: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{b, c\}\}$
Closed sets are $\emptyset, X, \{a\}, \{a, b\}$
$(r^*g^*)^*$-open sets are $\emptyset, X, \{b\}, \{c\}, \{b, c\}$
$(r^*g^*)^{**}$-closed sets are $\emptyset, X, \{a\}, \{a, b\}, \{a, c\}$.
gp-closed sets are $\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}$.
Here $\{b\}$ is gp-closed but not $(r^*g^*)^{**}$-closed.

Proposition 3.10: Every $(r^*g^*)^{**}$-closed set is g^{**}-closed set.

Proof: Let $A \subseteq U$, where U is g^*-open.
Since g^*-open implies$(r^*g^*)^*$-open. We have $cl(A) \subseteq U$.
Therefore A is g^{**}closed.
The converse need not be true as seen from the following example.

Example 3.11: Let $X = \{a, b, c\}$; $\tau = \{\emptyset, X, \{a\}\}$
Closed sets of (X, τ) are $\emptyset, X, \{b, c\}$,
$g **$-closed sets are $\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}$
$(r^*g^*)^{**}$-closed sets are $\emptyset, X, \{b, c\}$,
$\{b\}, \{c\}, \{a, b\}, \{a, c\}$ are g**-closed sets but not $(r^*g^*)^{**}$-closed sets.

Theorem 3.12: The union of two $(r^*g^*)^{**}$-closed sets is $(r^*g^*)^{**}$-closed.

Proof: Let A and B be $(r^*g^*)^{**}$-closed sets.
To prove: AUB is $(r^*g^*)^{**}$closed.
Suppose $A \cup B \subseteq U$, where U is a $(r^*g^*)^*$-open set.
Then $A \subseteq U$ and $B \subseteq U$.
$\Rightarrow cl(A) \subseteq U$ and $cl(B) \subseteq U. \Rightarrow cl(A \cup B) = cl(A) \cup cl(B) \subseteq U$.
$\Rightarrow A \cup B$ is $(r^*g^*)^{**}$closed.

Proposition 3.13: The intersection of two $(r^*g^*)^{**}$ closed sets is a $(r^*g^*)^{**}$closed set.

Proof: Let A and B be two $(r^*g^*)^{**}$-closed sets.
Let $A \bigcap B \subseteq U$, where U is a $(r^*g^*)^*$-open set.
Let $A \subseteq U_1$ and $B \subseteq U_2$ whenever U_1 and U_2 are $(r^*g^*)^{**}$-open sets. Then
$\Rightarrow cl(A) \subseteq U_1$ and $cl(B) \subseteq U_2$
Now $cl\left(A \bigcap B\right) = cl(A) \bigcap cl(B) \subseteq U_1 \cap U_2$ whenever $U_1 \cap U_2$ is $(r^*g^*)^*$-open.
$\Rightarrow A \cap B$ is a $(r^*g^*)^{**}$-closed set.

Example 3.14: Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$
Closed sets are $\emptyset, X, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \{d\}$
$(r^*g^*)^{**}$-closed sets are $\emptyset, X, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}$

Here $\{a, c, d\} \cap \{a, b, d\}$={a} which is $(r^*g^*)^{**}$-closed

Proposition 3.15: If A is both $(r^*g^*)^*$open and $(r^*g^*)^{**}$-closed, then A is closed.

Proof: Let A be $(r^*g^*)^*$-open set in X and also $(r^*g^*)^{**}$-closed set in X
Since A is $(r^*g^*)^*$-open, take $U = A \Rightarrow cl(A) \subseteq U = A \Rightarrow cl(A) \subseteq A$
But $A \subseteq cl(A) \Rightarrow Cl(A) = A$. Hence A is closed.

Proposition 3.16: If A is $(r^*g^*)^{**}$ closed set of (X, τ), such that A$\subseteq B \subseteq cl(A)$, then B is also a $(r^*g^*)^{**}$-closed set of (X, τ).

Proof: Let $B \subseteq U$ and U is $(r^*g^*)^*$-open set. Since $A \subset B \subset cl(A)$,
we have $A \subset B \subset U$ and since A is $(r^*g^*)^{**}$-closed, $cl(A) \subseteq U$. But $B \subset cl(A) \Rightarrow cl(B) \subset cl(A) \subset U$
$\Rightarrow cl(B) \subseteq U$whenever $B \subset U$ and U is $(r^*g^*)^{**}$open..
$\Rightarrow$ B is$(r^*g^*)^{**}$-closed

4. (r*g*) continuous maps**

Definition 4.1: A map $f : (X, \tau) \to (Y, \sigma)$ is called a (r*g*)**-continuous if $f^{-1}(V)$ is a (r*g*)**-closed set of (X, τ) for every closed set V of (Y, σ).

Proposition 4.2: Every continuous map is $(r^*g^*)^{**}$-continuous but the converse need not be true.

Example 4.3: Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{c\}, \{a, c\}\}$.
Closed sets are $\emptyset, X, \{a, b\}, \{b\}$
$(r^*g^*)^{**}$-closed sets are $\emptyset, X, \{b\}, \{a, b\}, \{b, c\}$
Let $\sigma = \{\emptyset, Y, \{c\}\}$. Closed sets are $\emptyset, Y, \{a, b\}$,
Define a function $f: (X, \tau) \to (Y, \sigma)$ by $f(a) = c; f(b) = a; f(c) = b$.
$f^{-1}(\{a, b\}) = \{b, c\}$ is (r*g*)**closed but not closed in (X, τ).Hence f is(r*g*)**-continuous but not continuous.

Proposition 4.4: Every $(r^*g^*)^{**}$-continuous map is gs-continuous function but the converse need not be true.

Example 4.5: Let $X = Y = \{a, b, c\}$ $\tau = \{\emptyset, X, \{c\}, \{a, c\}\}$ $\sigma = \{\emptyset, Y, \{b, c\}\}$
σ Closed sets are $\emptyset, Y, \{a\}$
$(r^*g^*)^{**}$-closed sets are $\emptyset, X, \{b\}, \{a, b\}, \{b, c\}$
gs-closed sets are $\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}$
Define a function $f: X \to Y$ by $f(a) = b; f(b) = c; f(c) = a$
$f^{-1}(\{a\}) = \{c\}$ is gs closed in (X, τ).
Which implies that f is gs-continuous but $\{c\}$ is not $(r^*g^*)^{**}$-closed in (X, τ). Therefore f is not $(r^*g^*)^{**}$-continuous.

Proposition 4.6: Every $(r^*g^*)^{**}$-continuous map is αg-continuous map but the converse need not be true.

Example 4.7: Let $X = Y = \{a, b, c\}$ $\tau = \{\emptyset, X, \{a\}\}$; $C = \{\emptyset, Y, \{a, c\}\}$;
$\sigma = \{\emptyset, Y, \{a, c\}\}$.
σ Closed sets are $\emptyset, Y, \{b\}$. $(r^*g^*)^{**}$-closed sets are $\emptyset, X, \{b\}, \{a, b\}, \{b, c\}$
αg-closed sets are $\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}$
Define a function $f: X \to Y$ by $f(a) = a; f(b) = c; f(c) = b$
$f^{-1}(\{b\}) = \{c\}$, αg-closed in (X, τ).
Which implies that f is αg-continuous but $\{c\}$ is not $(r^*g^*)^{**}$-closed in (X, τ). Therefore f is not $(r^*g^*)^{**}$-continuous.

Proposition 4.8: Every $(r^*g^*)^{**}$-continuous map is gp-continuous map but the converse need not be true.

Example 4.9: Let $X = Y = \{a, b, c\}$
$\tau = \{\emptyset, X, \{c\}, \{b, c\}\}$; $\sigma = \{\emptyset, Y, \{a, c\}\}$.
Define a function $f: X \to Y$ by $f(a) = a; f(b) = c; f(c) = b$
$f^{-1}(\{c\}) = \{b\}$, gp-closed in (X, τ).
Which implies that f is gp-continuous but $\{b\}$ is not $(r^*g^*)^{**}$-closed in (X, τ). Therefore f is not $(r^*g^*)^{**}$-continuous.

Theorem 4.10: Composition of two $(r^*g^*)^{**}$-continuous maps need not be $(r^*g^*)^{**}$-continuous.

Example 4.11: Let $X = \{a, b, c\}$; $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$
Closed sets are $\emptyset, X, \{a, c\}, \{c\}, \{a\}$
$(r^*g^*)^*$-open sets are $\emptyset, X, \{b, c\}, \{a, b\}, \{b\}, \{c\}, \{a\}$
$(r^*g^*)^{**}$-closed sets are $\emptyset, X, \{a\}, \{c\}, \{a, c\}$
Let $Y = \{a, b, c\}$; $\sigma = \{\emptyset, Y, \{c\}, \{b, c\}\}$
Closed sets are $\emptyset, Y, \{a, b\}, \{a\}$
$(r^*g^*)^*$-open sets are $\emptyset, Y, \{b, c\}, \{c\}, \{b\}$
$(r^*g^*)^{**}$-closed sets are $\emptyset, Y, \{a\}, \{a, b\}, \{a, c\}$
Define $f: (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = a; f(c) = b; f(b) = c$
Now $f^{-1}(\{a, b\}) = \{a, c\}$ and $f^{-1}(\{a\}) = \{a\}$, which are $(r^*g^*)^{**}$-closed in (X, τ). Therefore f is $(r^*g^*)^{**}$ is continuous.
Let $g: (Y, \tau) \to (Z, \eta)$ where $Z = \{a, b, c\}$; $\eta = \{\emptyset, X, \{c\}\}$
Closed sets are $\emptyset, Z, \{a, b\}$.
Let $g(a) = a; g(b) = c; g(c) = b$
$g^{-1}(\{a, b\}) = \{a, c\}$, which is $(r^*g^*)^{**}$-closed in $Y \Rightarrow g$ is $(r^*g^*)^{**}$ is continuous.
But $(g \circ f)^{-1}(\{a, b\}) = f^{-1}(g^{-1}\{a, b\})$

$$= f^{-1}\{(a, c)\}$$
$$= \{a, b\}$$

Which is not $(r^*g^*)^{**}$-closed set in (X, τ) which implies that the composition of two $(r^*g^*)^{**}$-continuous maps is not $(r^*g^*)^{**}$-closed.

Definition 4. 12: A function $f:(X,\tau) \rightarrow (Y,\sigma)$ is called a **(r*g*)**-irresolute** map if $f^{-1}(V)$ is a (r*g*)**-closed set of (X,τ) for every (r*g*)**-closed set V of (Y,σ).

Example 4.13: Let $X = Y = \{a, b, c\}$
$\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$
$(r^*g^*)^{**}$-closed sets $X, \emptyset, \{a\}, \{c\}, \{a, c\}$
$\sigma = \{\emptyset, Y, \{a\}\}$
$(r^*g^*)^{**}$-closed sets are $\emptyset, Y, \{b, c\}$
Define $f(a) = b, f(c) = c, f(b) = a$

$$f^{-1}(\{b, c\}) = \{a, c\}$$

$\{a, c\}$ is $(r^*g^*)^{**}$-closed sets in (X, τ)
Therefore f is a $(r^*g^*)^{**}$-irresoute mapping.
The following theorem gives some properties of $(r^*g^*)^{**}$-irresolute map.

Theorem 4.14:
1. Every $(r^*g^*)^{**}$-irresolute map is $(r^*g^*)^{**}$-continuous.
2. Every $(r^*g^*)^{**}$-irresolute map is gs-continuous.
3. Every $(r^*g^*)^{**}$-irresolute map is αg-continuous.
4. Every $(r^*g^*)^{**}$-irresolute map is gp-continuous.
5. Every $(r^*g^*)^{**}$-irresolute map is g^{**}-continuous.

Example 4.15: The converse of the above theorems need not be true.
1. Let $X = \{a, b, c\}$; $\tau = \{X, \emptyset, \{b\}, \{a, b\}, \{b, c\}\}$
   Closed sets are $X, \emptyset, \{a, c\}, \{c\}, \{a\}$
   $(r^*g^*)^{**}$-closed $\emptyset, X, \{a\}, \{c\}, \{a, c\}$
   $Y = \{a, b, c\}$; $\sigma = \{Y, \emptyset, \{c\}, \{a, c\}\}$
   Closed sets are $Y, \emptyset, \{b\}, \{a, b\}$
   $(r^*g^*)^{**}$-closed $\emptyset, Y, \{b\}, \{a, b\}, \{b, c\}$
   Define f: $(X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = a, f(b) = c, f(c) = b$
   $f^{-1}(\{a, b\}) = \{a, c\}, f^{-1}(\{b\}) = \{c\}$ are $(r^*g^*)^{**}$-closed.
   Therefore f is $(r^*g^*)^{**}$-continuous.
   Now $f^{-1}(\{b, c\}) = \{b, c\}$ which is not $(r^*g^*)^{**}$-closed.
   Which implies that f is not $(r^*g^*)^{**}$-irresolute.
   Here f is $(r^*g^*)^{**}$-continuous but not irresolute.
2. Let $X = \{a, b, c\}$; $\tau = \{\emptyset, X, \{c\}, \{a, c\}\}$
   $Y = \{a, b, c\}$; $\sigma = \{\emptyset, Y, \{a, c\}\}$
   Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = a; f(b) = a;\ f(c) = b$
   Here f is gs-continuous but not $(r^*g^*)^{**}$-irresolute.
3. Let $X = \{a, b, c\}$; $\tau = \{\emptyset, X, \{a\}\}$
   $Y = \{a, b, c\}$; $\sigma = \{\emptyset, Y, \{a, b\}, \{b\}\}$
   Let f: $(X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = a; f(b) = c;\ f(c) = b$
   Here f is αg-continuous but not $(r^*g^*)^{**}$-irresolute.
4. Let $X = \{a, b, c\}$; $\tau = \{\emptyset, X, \{c\}, \{b, c\}\}$
   $Y = \{a, b, c\}$; $\sigma = \{\emptyset, Y, \{a, b\}\}$

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b; f(b) = c;\ f(c) = a$
Here f is gp-continuous but not $(r^*g^*)^{**}$-irresolute.

5. Let $X = \{a, b, c\};\ \tau = \{\emptyset, X, \{c\}, \{b, c\}\}$
   $Y = \{a, b, c\};\ \sigma = \{\emptyset, Y, \{a, b\}\}$
   Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = b; f(b) = c;\ f(c) = a$
   Here f is g^{**}-continuous but not $(r^*g^*)^{**}$-irresolute.

Remark 4.16: $(r^*g^*)^{**}$-irresoluteness is independent of gs-irresoluteness, αg-irresoluteness, gp-irresoluteness and g^{**}-irresoluteness.

Proposition 4.17: Composition of $(r^*g^*)^{**}$-irresolute maps is again an $(r^*g^*)^{**}$-irresolute map.

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