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Reliability Approximation of Weibull Density with Squared Error Loss Function

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Abstract
Weibull distribution and failure censoring (type I) play an important role in life testing and reliability engineering. The failure censoring (type I) can improve the efficiency of test by allowing testers to assign a pre-assigned number of units to different test facilities. In this paper, we have obtained the bayes estimator of reliability as well as the Approximate Bayes Estimator of two parameter Weibull population by Lindley Approximation Method under Squared error loss function (SQELF). A numerical comparison is done and is found that proposed approximate Bayes estimator of Reliability function perform better than ML Estimator.

Keywords: Reliability Function, Failure Censoring, Weibull Density, Lindley Approximation, Bayesian Technique, Squared Error Loss.

Introduction.

A useful general distribution for describing failure time data is the Weibull distribution. The distribution is named after the Swedish professor Waloddi Weibull, who demonstrated the appropriateness of this distribution for modeling a wide variety of different data sets (Hahn and Shapiro, 1967).Weibull (1951) showed that the distribution is also useful in describing the wear out of fatigue failures. Estimation and properties of the Weibull distribution is studied by many authors [see Kao (1959)].
This distribution is particularly useful in reliability work since it is a general distribution which, by adjustment of the distribution parameters, can be made to model a wide range of life distribution characteristics of different classes of engineered items. ; for example, the Weibull distribution has been used to model the life times of electronic components, relays, ball bearings, or even some businesses).Weibull distribution has been extensively used in life testing and reliability probability problems.
The probability density function of Weibull distribution with $'\nu'$ as scale and 'μ' as shape parameters is given as

$$f(x;\nu,\mu) = \mu\nu x^{(\mu-1)} \exp(-\mu\, x^{\mu}) \quad ; \quad x,\nu,\mu > 0 \qquad (1.1)$$

$$F(x;\nu,\mu) = 1 - \exp(-\mu\, x^{\mu}) \quad ; \quad x,\nu,\mu > 0 \qquad (1.2)$$

The reliability function is the complement to the cumulative distribution function (i.e., R(t)=1-F(t)); the reliability function is also sometimes referred to as the survivorship or survival function since it describes the probability of not failing or of surviving until a certain time **t**.The Reliability Function of Weibull Distribution is given as

$$R(t) = exp(-\nu t^{\mu}) \quad ; \quad t > 0 \qquad (1.3)$$

$$H(t) = \mu\nu t^{(\mu-1)} \quad ; \quad t > 0 \qquad (1.4)$$

Censoring

In most studies of product reliability, not all items in the study will fail. In other words, by the end of the study the researcher only knows that a certain number of items have not failed for a particular amount of time, but has no knowledge of the exact failure times (i.e., "when the items would have failed"). Those types of data are called censored observations. The issue of censoring, and several methods for analyzing censored data sets, are also described in great detail in the context of the Survival Analysis module. Censoring can occur in many different ways.

Type I and II censoring

So-called Type I censoring describes the situation when a test is terminated at a particular point in time, so that the remaining items are only known not to have failed up to that time. In this case, the censoring time is often fixed, and the number of items failing is a random variable. In Type II censoring the experiment would be continued until a fixed proportion of items have failed. In this case, the number of items failing is fixed, and time is the random variable.

Squared error loss function (SQELF)

The most widely used loss function in estimation problems is quadratic loss function given as $L(\hat{\theta},\theta) = k(\hat{\theta}-\theta)^2$ where $\hat{\theta}$ is the estimate of θ, the loss function is called quadratic weighed loss function if k=1, we have

$L(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2$(1.5)

known as squared error loss function (SQELF). This loss function is symmetrical because it associates the equal importance to the losses due to overestimation and under estimation with equal magnitudes however in some estimation problems such an assumption may be inappropriate. Overestimation may be more serious than underestimation or Vice-versa Ferguson(1985). Canfield (1970), Basu and Ebrabimi(1991). Zellner (1986) Soliman (2000) derived and discussed the properties of varian's (1975) asymmetric loss function for a number of distributions.

In a Bayesian setup, the unknown parameter is viewed as random variable. The uncertainty about the true value of parameter is expressedby prior distribution. The parametric inference is made using the posterior distribution which is obtained by incorporating the observed data into the prior distribution using the Bayes theorem, the first theorem of inference. Hence, we update the prior distribution in the light of observed data. Thus, the uncertainty about the parameter prior to the experiment is represented by the prior distribution and the same after the experiment is represented by the posterior distribution(Berger(1980)).

The Estimators

Let $x_1, x_2, \ldots\ldots\ldots x_n$ be the life times of 'n' items that are put on test for their lives, follow a weibull distribution with density given in equation (1.1). The failure times are recorded as they occur until a fixed number 'r' of times failed(Type I Censoring). Let $= (x_{(1)}, x_{(2)}, \ldots\ldots\ldots\ldots\ldots, x_{(n)})$, where $x_{(i)}$ is the life time of the ith item. Since remaining (n-r) items yet not failed thus have life times greater than $x_{(r)}$.

The likelihood function can be written as

$L(x|\nu,\mu) = \frac{n!}{(n-r)!}(\mu\nu)^r \prod_{i=1}^{r} x_i^{(\mu-1)} \exp(-\delta\nu)$,(2.1)

Where$\delta = \sum_{i=1}^{r} x_i^{\mu} + (n-r)x_r^{\mu}$

The logarithm of the likelihood function is

$\log L(x|\nu,\mu) \propto r\, log\, \mu + r \log \nu + (\mu-1)\sum_{i=1}^{r} \log x_i - \delta\nu$,(2.2)

assuming that 'μ' is known, the maximum likelihood estimator $\hat{\nu}_{ML}$of ν can be obtain by using equation (2.2) as

$\hat{\nu}_{ML} = r/\delta$(2.3)

In case if both the parameters μ and ν are unknown their MLE's $\hat{\mu}_{ML}$and $\hat{\nu}_{ML}$ can be obtained by solving the following equation

$\frac{\delta}{\delta \nu}\log L = \frac{r}{\nu} - \delta = 0$, (2.4a)

$\frac{\delta \log L}{\delta \mu} = \frac{r}{\mu} + \sum_{i=1}^{r} \log x_i - \nu\delta_1 = 0$,(2.4b)

where

$\delta_1 = \sum_{i=1}^{r} x_i^{\mu} \log x_i + (n-r)x_r^{\mu} \log x_r$, eliminating ν between the two equations of (2.4) and simplifying we get

$\hat{\mu}_{ML} = \frac{r}{\delta^*}$(2.5)

Where $\delta^* = \left[\frac{r\delta_1}{\delta} - \sum_{i=1}^{r} log x_i\right]$

Equation (2.5) may be solved for Newton-Raphson or any suitable iterative Method and this value is substituted in equation (2.4b) by replacing with μ get $\hat{\mu}$ as

$\hat{\nu}_{ML} = \frac{\frac{r}{\hat{\mu}_{ML}} + \sum_{i=1}^{r} log x_i}{\sum_{i=1}^{r} x_i^{\hat{\mu}_{ML}} \; log x_i + (\mathrm{n-r}) x_r^{\hat{\mu}_{ML}} \; log x_r}$,(2.6)

Bayes Estimator of Scale Parameterν when shape Parameter μ is known :

If μ is known assume gamma prior $\phi(c,d)$ as cojugate prior for ν as

$\phi(\nu|\underline{x}) = \frac{d^c}{\Gamma c}(\mathrm{v})^{(c+1)} \exp(-\mathrm{dv})\,; (c,d) > 0, \nu > 0$,(3.1)

The posterior distribution of ν using equation (2.1) and (3.1) we get

$\psi(\theta|\underline{x}) = \frac{(\delta+d)^{r+c}}{\Gamma(r+c)}(\mathrm{v})^{(r+c-1)} \exp\big(-\mathrm{v}(\delta+d)\big)$, (3.2)

Under squared error loss function, the Bayes estimator $\hat{\nu}_{BSQ}$, is the posterior mean given by

$\hat{\nu}_{BSQ} = \frac{(r+c)}{(\delta+d)}$(3.3)

Bayes Estimator of R(t)

The posterior distribution of R using equation (1.3) and (3.2), is given as

$h(R|t) = \frac{[\,\omega\,(\delta+d)]^{(r+c)}}{\Gamma(r+c)}(-\log R)^{(r+c-1)} R^{(\,\mathrm{c}\,(\delta+d)\,-\,1)} dR$;(3.4)

Where $\omega = t^{-d}$

The Bayes estimator of R(t) under squared error loss function using (3.4) is given by

$\hat{R}_{BSQ} = \left[1 + \frac{1}{\omega\,(\delta+d)}\right]^{-(r+c)}$; (3.5)

4.The Bayes estimators with ν and μ unknown

The joint prior density of ν and μ is given by

$\phi^*(\nu|p) = \phi_1(\nu|\mu).\phi_2(\mu)$

$\phi^*(\nu|\mu) = \frac{1}{\lambda\Gamma\xi} p^{-\xi} \mathrm{v}^{(\xi-1)}.\exp\left[-\left(\frac{\nu}{\mu} + \frac{\mu}{\lambda}\right)\right]$; $(\nu,\mu,\lambda,\xi) > 0$, (4.1)

where

$\phi_1(\nu|\mu) = \frac{1}{\Gamma\xi}\nu^{-\xi}\mathrm{v}^{(\xi-1)}.\exp\left[-\frac{\nu}{\mu}\right]$;(4.2)

And

$\phi_2(\mu) = \frac{1}{\lambda}\exp\left(-\frac{\mu}{\lambda}\right)$; (4.3)

The joint posterior density of ν and μ is

$\psi^*(\nu,\mu|\underline{x}) = \frac{\frac{1}{\lambda\Gamma\xi} p^{-\lambda} \mathrm{v}^{(\xi+1)} \exp\left[-\left\{\frac{\nu}{\mu}+\frac{\mu}{\lambda}\right\}\right](\mathrm{v}\mu)^r \prod_{i=1}^{r} x_i^{(\mu-1)} e^{-\mu\nu}}{\iint \frac{1}{\lambda\Gamma\xi}\mu^{(r-\xi)}\mathrm{v}^{(r+\xi+1)} \prod_{i=1}^{r} x_i^{(\mu-1)}.\exp\left[-\left\{\frac{\nu}{\mu}+\frac{\mu}{\lambda}+\mu\mathrm{v}\right\}\right] d\nu d\mu}$; (4.4)

Approximate Bayes Estimators

The Bayes estimators of a function $\rho = \rho(\nu,\mu)$ of the unknown parameter $\nu\ and\ \mu$ under squared error loss is the posterior mean

$\hat{\varrho}_{ABS} = E(\mu|\underline{x}) = \frac{\iint \phi(\nu\mu)\phi^*(\nu,\mu|\underline{x})d\nu d\mu}{\iint \phi^*(\nu,\mu|\underline{x}).d\nu d\mu}$;(4.5)

By using Lindley approximation method we evaluate equation (4.5)

$E(\varrho(\nu,\mu)|\underline{x}) = \frac{\int \phi(\nu).e^{(l(\nu)+\varrho(\nu))}d\nu}{\int e^{(l(\nu)+\varrho(\nu))}d\nu}$;(4.6)

Where $l(\nu) = \log\phi(\nu)$, and $\phi(\nu)$ is an arbitrary function of ν and $l(\nu)$ is the logarithm likelihood function

The Lindley approximation (Lindley (1980)) for two parameter is

$E(\hat{\varrho}(\nu,\mu)|\underline{x}) = \varrho(\nu,\mu) + \frac{A}{2} + \rho_1 A_{12} + \rho_2 A_{21} + \frac{1}{2}[l_{30}B_{12} + l_{21}C_{12} + l_{12}C_{21} + l_{03}B_{21}]$, (4.7)

Where

$A = \sum_1^2\sum_1^2 \varrho_{ij}\sigma_{ij}$; $l_{\eta\epsilon} = (\delta^{(\eta+\epsilon)}l|\delta\nu_1^{\eta}\delta\nu_2^{\epsilon})$; where$(\eta+\epsilon) = 3$, for $i,j = 12$; $\varrho_i = (\delta\varrho|\delta\nu_i)$;

$\varrho_i = \frac{\delta\varrho}{\delta\nu_i}$; $\varrho_{ij} = \frac{\delta^2\varrho}{\delta\nu_i\delta\nu_j}$; $\forall i \neq j$;

$A_{ij} = \varrho_i\sigma_{ij} + \varrho_j\sigma_{ji}$; $B_{ij} = (\varrho_i\sigma_{ii} + \varrho_j\sigma_{ij})\sigma_{ii}$; $C_{ij} = 3\varrho_i\sigma_{ii}\sigma_{ij} + \varrho_j(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2)$;

Where σ_{ij} is the (i,j)th element of the inverse of matrix $\{-l_{jj}\}; i,j = 1,2$ s.t. $l_{ij} = \frac{\delta^2 l}{\delta\nu_i\delta\nu_j}$.

All the function in above equationsis evaluated at MLE of (ν_1,ν_2).In our case $(\nu_1\nu_2) = (\nu,\mu); So\ \phi(\nu) = \phi(\nu,\mu)$

To apply Lindley approximation (4.5), we first obtain σ_{ij} , elements of the inverse of $\{-l_{jj}\}; i,j = 1,2$,which can be shown to be

$\sigma_{11} = \frac{M}{D}$, $\sigma_{12} = \sigma_{21} = \frac{\delta_1}{D}$,$\sigma_{22} = \frac{r}{D\,\theta^2}$,(4.8a)

Where$M = \left(\frac{r}{\mu^2} + \nu\delta_2\right)$;$D = \left[\frac{r}{\nu^2}\left(\frac{r}{\mu^2} + \nu^2\delta_2\right)\right]$; (4.8b)

$\delta_2 = \sum_{i=1}^{r} x_i^{\mu}(logx_i)^2 + (n-r)x_r^{\mu}(logx_r)^2$; (4.8c)

To evaluateρ_i , take the joint prior $\phi^*(\nu|\mu)$

$\phi^*(\nu|\mu) = \frac{1}{\lambda\Gamma\xi}\mu^{-\xi}\nu^{(\xi-1)}.\exp\left[\left\{-\frac{\nu}{\mu} + \frac{\mu}{\lambda}\right\}\right]$; $(\nu,\mu,\lambda,\xi) > 0$, (4.9)

$\Rightarrow \rho = \log[\phi^*(\nu|\mu)] = constant - \xi log\mu - (\xi-1)log\nu - \frac{\nu}{\mu} - \frac{\mu}{\lambda}$

Therefore

$\rho_1 = \frac{\partial\rho}{\partial\nu} = \frac{(\xi-1)\nu}{\nu} - \frac{1}{\nu}$;(4.9a)

and

$\rho_2 = \frac{\nu}{\mu^2} - \frac{1}{\lambda} - \frac{\xi}{\mu}$;(4.9b)

Further more

$l_{21} = 0$; $l_{12} = -\delta_2$; $l_{03} = \frac{2r}{p^3} - \nu\delta_3$; (4.9c)

$and\ \ l_{30} = \frac{2r}{\nu^3}$; (4.9d)

Where $\delta_3 = \sum_{i=1}^{r} x_i^{\nu}(logx_i)^3 + (n-r)x_r^{\nu}(logx_r)^3$

By substituting above values in eqn. (4.7), yields the Bayes estimator under SELF using Lindley approximation denoted by $\hat{\varrho}_{ABS}$

$\hat{\varrho}_{ABSQ} = E(\varrho(\nu,\mu)) = \varrho(\nu,\mu) + U + \varrho_1 U_1 + \varrho_2 U_2$; (4.10)

Where U $= \frac{1}{2}[\varrho_{11}\sigma_{11} + \varrho_{21}\sigma_{21} + \varrho_{12}\sigma_{12} + \varrho_{22}\sigma_{22}]$;(4.10a)

$U_1 = \frac{1}{\nu^2D^2}[\frac{M\nu D}{\mu}(\mu(\xi-1)-1) + \frac{\nu^2\delta_1 D}{\lambda\mu^2}\{\lambda\nu - \mu^2 - \lambda\xi\mu\}$

$+ \frac{rM^2}{\nu} - \frac{rM\delta_1}{2} - \nu^2\delta_1^2\delta_2 + \frac{r^2}{\nu^3}\delta_1 - \frac{\nu r\delta_1\,\delta_3}{2}]$; (4.10b)

$U_2 = \frac{1}{\nu^2D^2}[\frac{\nu\ \ \delta_1 D}{\mu}(\mu(\xi-1)-\nu) + \frac{rD}{\lambda\mu^2}\{\lambda\nu - \mu^2 - \lambda\xi\mu\}$

$+ \frac{rM\delta_1}{\nu} - \frac{3\delta_1 r\delta_2}{2} + \frac{r^2}{\nu^2\mu^3} - \frac{r^2\delta_3}{2\nu}]$; (4.10c)

All the function of right hand side of the equation (4.10) are to be evaluated for $\hat{v}_{ML}$ and $\hat{\mu}_{ML}$.

Approximate Bayes Estimates of Reliability Under Squared Error Loss function

with equations(4.10)-(4.10c), the different Approximate Bayes estimators Under SQELF using Lindley's approximation given by
Special cases:

(i) substituting $\phi(\nu,\mu) = R = e^{-\nu}t^{\mu}$ in equation(4.7), we get the Approximate Bayes Estimator of Reliability R=R(t) as

$$\hat{R}_{ABSQ} = R\left[1 + \frac{Rt^{\mu}}{2D}\xi_1 - t^{\mu}(U_1 + \nu \log t\, U_2)\right] \;\; ; \text{at } (\hat{v}_{ML}, \hat{\mu}_{ML}), \qquad (4.11)$$

where

$$\xi_1 = Mt^{\mu} + \log t\,(\nu\, t^{\mu} - 1) + \left[2\delta_1 + \frac{rlogt}{\nu}\right]$$

Numerical Calculations and Comparison

The numerical calculations are done by using R Language programming and results are presented in form of tables.

1. The values of $(\nu,\mu)$and are generated from the equations (4.2-4.3) for given c=2, and d=3, which comes out to be ν=0.238 and μ=0.227. For these values of ν and μ the Weibull random variates are generated.
2. Taking the different sizes of samples n=25 (25) 100 with failure censoring, MLE's, the Approximate Bayes estimators of Reliability,and their respective MSE's (in parenthesis) by repeating the steps 500 times, are presented in the table(1) forparameters of prior distribution c =2 , and d=3.
3. Table(1) presents the MLE of R(t) and Approximate Bayes estimators of reliability function R(t) of Weibull density under SELF (for ν and μ both unknown) with their respective MSE's. The all four estimators are efficient for larger sample size but as sample approaches to 100 their MSE's started increasing.

Table(1) Mean and MSE's of R(t)

$(\lambda = 2, \xi = 3, = 0.238, \mu = 0.227, t = 2, R(t) = .07568$

n	r	$\hat{R}_{ML}$	$\hat{R}_{BSQ}$	$\hat{R}_{ABSQ}$
25	20	0. 649642	0. 79958	0. 74576
		(1.7239x10^{-5})	**(3.0711x10^{-7})**	**(4.0544x10^{-5})**
50	30	0.713952	0. 800211	0. 750967
		(1.63169x10^{-7})	**(4.0081x10^{-7})**	**(3.67188x10^{-6})**
75	50	0. 740252	0. 793433	0. 755524
		(2.7167x10^{-8})	**(4.3171x10^{-8})**	**(4.0074x10^{-8})**
100	75	0. 76858	0. 8937546	0. 8482655
		(7.1028x10^{-6})	**(7.1229 x10^{-4})**	**(7.0887 x10^{-5})**

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