Nat. Volatiles & Essent. Oils, 2021; 8(5): 2143 - 2148

Singular Two - Point Boundary Value Problems With Spline Finite Difference Methods

D. Bhagyamma[1], Richa Gupta[2*], M.V. Ramana Murthy [3], K.Venkata Sai Vikramaditya[4]

[1] *Department of Mathematics, Maturi Venkata Subba Rao Engineering College, Nadargul, Hyderabad.*

[2*]*Department of Mathematics, Sarvepally Radhakrishnan University, Bhopal, Madhya Pradesh*

[3]*Department of Mathematics & Humanities, Mahatma Gandhi Institute of Technology, Gandipet, Hyderabad*

[4]*Stevens Institute of Technology, New Jersey*

***Email:** Richasharad.gupta@gmail.com*

Abstract:

The focus of this article would be to use various numerical techniques and discover the numerical performance of singular two-point boundary value problems. This is certainly done in a comparative manner, mostly by using the application of finite element methods. The calculated solutions have been discussed extensively in order to fully understand the conduct of the physical processes represented by the replica equation. We propose a method for resolving a class of linear and non - linear singular two-point boundary value problems that is both effective and efficient. The outcome reveal that the method is exceedingly successful straightforward, and simple.

Keywords:

Boundary Layer, Two-Point Boundary Problem, Finite element method, Physical behavior.

Introduction

Singular two-point boundary value problems are encountered in many physical models such as electro hydrodynamics and some thermal explosions, and thus, have been investigated by using a variety of numerical methods [1-5].

In most cases, it is not possible to solve the singular boundary value problems analytically. However, there are some numerical/approximate methods used in the literature, for instance, finite difference methods [6- 10].

A novel approach that combines a modified decomposition method with the cubic B-spine collocation technique is presented in finite element methods [11-15] to obtain approximate solution with high accuracy. In order to avoid solving such nonlinear algebraic or transcendental equations for two-point boundary value problems, in [16], extended ADM was introduced for nonsingular problems with boundary conditions.

Herein, an important instance also is the use of an automatic plotter that frequently requires interpolation at great many intermediate points. However, it is well known since then that the cubic spline method of Bickley gives only second order convergent approximations. But cubic spline itself is a fourth order process depicted below. Errors are shown in Table-1.

Mathematical Details of the Discretization

$$A^{-\alpha}(A^{\alpha}, I')' = f(A, I) \qquad 0 < A \le 1, \qquad (5.1)$$

$$I(0) = A, \qquad I(1) = B,$$

$$(A, I) \in \{[0,1 \times R]\}; (A) f(A, I)\frac{\partial f}{\partial I}, \qquad \frac{\partial f}{\partial I} \ge 0$$

$$0 = A_0 < A_1 < A_2 \cdots\cdots\cdots\cdots\cdots\cdots + < A_N = 1.$$

$I_j = I(A_j)$ and $f_j = f(A_j, I_j)$

$$A^{-\alpha}\frac{d}{dA}\left(A^{\alpha}\frac{dI}{dA}\right) = \frac{C_{j-1}}{h_j}(I_j - I) + \frac{C_j}{h_j}(I - I_{j-1}), \quad A_{j-1} < A < A_j \qquad (5.2)$$

$$h_j = A_j - A_{j-1} \quad \sigma_j = \frac{h_{j+1}}{h_j}$$

$[A^{-\alpha}(A^{\alpha}, I')']A = A_j = C_j = f_j$

$[A^{-\alpha}(A^{\alpha}, I')']A = A_{j-1} = C_{j-1} = f_{j-1}$

$$I(A_{j-1}) = I_{j-1}, \qquad I(A_j) = I_j,$$

$$I(A) = -\frac{B_j}{a}\left[(I_j A_{j-1}^a - I_{j-1}A_j^a) - A^a(I_j - I_{j-1})\right] + \left[B_j^* A^2\{2A(1+\alpha) - 3(2+\alpha)A_{j-1}\} + \frac{a_j}{a}A^a + a_j^*\right]C_j + \left[B_j^* A^2\{3(2+\alpha)A_j - 2A(1+\alpha)\} + \frac{b_j}{a}A^a + b_j^*\right]C_{j-1} \qquad (5.3)$$

$$A_{j-1} < A < A_j$$

$$B_j = \frac{a}{A_j^a - A_{j-1}^a}, \qquad B_j^* = \frac{1}{6h_j(1+\alpha)(2+\alpha)}$$

$$a_j = -B_j B_j^*\left[\alpha A_j^2(2A_j - 3A_{j-1}) + 2A_j^2(A_j - 3A_{j-1}) + A_{j-1}^3 + (\alpha + 4)\right]$$

$$b_j = -B_j B_j^*\left[A_{j-1}^3(\alpha + 4) - \alpha A_{j-1}^2(3A_j - 2A_{j-1}) - 2A_{j-1}^2(3A_j - A_{j-1})\right]$$

$$a_j^* = \frac{B_j B_j^*}{a}\left[A_j^2 A_{j-1}^a\{2(1+\alpha)A_j - 3(2+\alpha)A_{j-1}\} + (\alpha+4)A_{j-1}^3 A_j^a\right]$$

$$b_j^* = \frac{B_j B_j^*}{a}\left[(\alpha+4)A_j^3 A_{j-1}^a - A_j^a A_{j-1}^2\{3(2+\alpha)A_j - 2(1+\alpha)A_{j-1}\}\right]$$

$$-B_j I_{j-1} + (B_j + B_{j+1})I_j - B_{j+1}I_{j-1} = A_j C_{j+1} + B_j C_j + C_j C_{j-1}$$

$$j = 1,2,3, \ldots\ldots., N-1 \qquad (5.4)$$

$$A_j = -6B^*_{j+1}A_j^{2+\alpha} + a_{j+1}$$

$$C_j = -6B^*_j A_j^{2+\alpha} - b_j$$

$$B_j = 6A_j^{1+\alpha}\left[(2+\alpha)\left(A_{j+1}B^*_{j+1} + A_{j-1}B^*_j\right) - (1+\alpha)A_j\left(B^*_{j+1} + B^*_j\right)\right] + \left(b_{j+1} - a_j\right)$$

$$C_{j-1} = f_{j-1}$$

$$-B_j I_{j-1} + \left(B_j + B_{j+1}\right)I_j - B_{j+1}I_{j-1} = A_j f_{j+1} + B_j f_j + C_j f_{j-1}$$

$$j = 1,2,3,\ldots\ldots, N-1 \qquad (5.5)$$

$$t\left(h_j\right) = \frac{1}{24}A_j^{\alpha}\left(1+\sigma_j^3\right)h_j^3 f_j'' + \cdots\cdots$$

$$BI + Cf + T = R \qquad (5.6)$$

$$I = \left(i_1, i_2, i_3, \cdots\cdots\cdots i_{N-1}\right)^T, \qquad T = \left(t_1, t_2, t_3, \cdots\cdots\cdots t_{N-1}\right)^T$$

$$f = \left(f_1, f_2, f_3, \cdots\cdots\cdots f_{N-1}\right)^T, R = \left((B_1X + Z_1f_0)O\cdots\cdots O(B_NY + X_Nf_N)\right)^T$$

$$B = \begin{pmatrix} B_1 + B_2 & -B_2 & 0 \\ -B_2 B_2 + B_3 & -B_3 & \\ & & \end{pmatrix}$$

$$-B_3 B_3 + B_4 - B_4$$

$$B_j > 0, X_j < 0, Y_j < 0, C_j < 0, C > 0$$

$$B\bar{I} + Cf(\bar{I}) = R \qquad (5.7)$$

$$(B + CF)E = T \qquad (5.8)$$

$E = \bar{I} - I$ and $EF = f(\bar{I}) - f(I)$

$$\left[\frac{\partial f_1}{\partial I_1}\frac{\partial f_2}{\partial I_2}\cdots\cdots\cdots\frac{\partial f_{N-1}}{\partial I_{N-1}}\right]$$

$$\frac{\partial f}{\partial I} \geq 0, \qquad F \geq 0 \qquad CF \geq 0$$

$$B + CF \geq B$$

$$(B + CF)^{-1} \leq B^{-1} \qquad (5.9)$$

$$\|E\| = \|(B + CF)^{-1}T\| \leqq \|B^{-1}\|\|T\| \qquad (5.10)$$

$$B^{-1} = \left(B^*_{i,j}\right), \quad a = 1 - \alpha. \qquad j = 1,2,3\cdots\cdots, i-1$$

$$B^*_{i,j} = \left(\frac{A_i}{A_1}\right)^* B^*_{1,j} - \frac{1}{a}\left(A_i^a - A_j^a\right), \qquad i = 1(1)N-1, i \geq j$$

$B^*_{1,j} = -\frac{1}{aD}\left[(A^a_N - A^a_{N-1})\left(A^a_{N-2} - A^a_j\right) - (A^a_N - A^a_{N-2})\left(A^a_{N-1} - A^a_j\right)\right]$ (5.11)

$$D = \left(\frac{A_{N-1}}{A_1}\right)^* (A^a_N - A^a_{N-2}) - \left(\frac{A_{N-2}}{A_1}\right)^* (A^a_N - A^a_{N-1})$$

$j = i, i+1, \dots\dots, N-1$

$B^*_{1,j} = \left(\frac{A_i}{A_1}\right)^* \quad B^*_{i,j}, i \leqq j$

$B^*_{1,j} = \frac{1}{aD_1}\left[\left(A^a_N - A^a_j\right)\left(A^a_j - A^a_{j-1}\right)\right]$ (5.12)

$$\sum_{j=1}^{i-1} B^*_{1,j} = -\frac{1}{a}\left(\frac{A_1}{A_N}\right)^a [(A^a_1 + A^a_2 + \cdots\cdots\cdots + A^a_{i-1}) - (i-1)A^a_N]$$

$$\sum_{j=1}^{N-1} B^*_{1,j} = \frac{1}{a}\left(\frac{A_1}{A_N}\right)^a \sum_{j=1}^{N-1}\left(A^a_N - A^a_j\right)$$

$\sum_{j=1}^{i-1} B^*_{i,j} = \frac{1}{a}\left[(N-i)A^a_i + \sum_{j=1}^{i-1} A^a_j - \left(\frac{A_i}{A_N}\right)^a \sum_{j=1}^{N-1} A^a_j\right]$ (5.13)

$$A_j = (jh)^{1/_{1-\alpha}}$$

$\sum_{j=1}^{i-1} B^*_{i,j} = \frac{1}{a}\left[(N-i+1)A^a_i + \sum_{j=1}^{i-1} A^a_j - \left(\frac{A_i}{A_N}\right)^a \sum_{j=1}^{N} A^a_j\right]$ (5.14)

$1^\lambda + 2^\lambda + \cdots\cdots\cdots\cdots (i-1)^\lambda < \frac{i^{\lambda+1}}{\lambda+1}$ (5.15)

$1^\lambda + 2^\lambda + \cdots\cdots\cdots\cdots N^\lambda < \frac{N^{\lambda+1}}{\lambda+1}$ (5.16)

$$\lambda = \frac{a}{1-\propto}$$

$\sum_{j=1}^{i-1} B^*_{i,j} < \frac{1}{a}\left[(N-i+1)A^a_i + \frac{i^{\lambda+1}h^\lambda}{\lambda+1} - \left(\frac{A_i}{A_N}\right)^a \frac{h^\lambda N^{\lambda+1}}{\lambda+1}\right]$ (5.17)

$< \frac{t^{\lambda+1}}{a(\lambda+1)h}$ (5.18)

$t = \left[\frac{(\lambda N+\lambda+1)a}{\lambda N(a+1-\propto)}\right]^{1/(1-\propto)}$

$$\|E\| < \frac{t^{\lambda+1}}{a(\lambda+1)h}\|t_j\|$$

$$\leq \frac{t^{\lambda+1}}{a(\lambda+1)h}\frac{A_j^{\propto}h^3}{12}|f''|$$

$$\leq \left(\frac{a}{\lambda+1}\right)^{\lambda+1} \frac{h^2 C}{a(\lambda+1)} \qquad (5.19)$$

Observations and Conclusion

Table 1 Absolute error ||E|| in problem 1

N	**||E||**
	α=0.5 σ=0.98 ρ=1/2
20	2.7 (-4)
40	8.3 (-5)
80	2.7 (-5)
100	2.7 (-6)

N	**||E||**
	α=0.5 σ=1.02 ρ=1/2
20	1.8 (-4)
40	4.2 (-5)
80	1.2 (-5)
100	2.4 (-6)

We can be used to solve a class of linear as well as nonlinear singular two-point boundary value problems. Note that for higher dimensional problems, the same discussion could be an important milestone in numerical modeling.

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