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Extrapolated Newton-Raphson method for solving functions of two dimensions.

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Abstract

The purpose of this study is to extrapolated Newton- Raphson method for solving functions of two variables . we compare the two different methods of solving functions of two variables. To describe the procedures and the techniques that can be followed in extrapolated Newton Raphson method for solving functions of two variables. To analyze the convergence of the proposed method. To explain the advantages of the proposed method over the existing methods. The intention of this study is to introduced a new method and compare its result with that of Newton-Raphson method. Thus, from discussion of the results, it is easily observed that Extrapolated Newton's method is the efficient the method to solve systems of nonlinear equations with two variables. One practical example of systems of nonlinear equations is considered. Analysis of the results shows that the Newton's method takes 4 iterations to converge and Extrapolated Newton's method converges after three iterations as per the error of tolerance considered.

Introduction

The major goal of numerical analysis is to find the approximation solution to difficult problem (or that cannot be solved analytically) by using different numerical techniques.One of the important methods to find approximation solution of numerical analysis is Newton-Raphson method. Newton –Raphson method, named after Isaac Newton and Joseph–Raphson is a method for finding successively better approximations to the roots (or Zeroes) of real valued function [9]. Solutions of equation by iteration are of interest to many science and engineering problems. For this purpose, several methods which require evaluation of one function and its derivatives at each steps of the iteration have an order of convergence two in many cases [4]. Iterative methods are often useful even for linear problems involving large number of variables, where direct method would be prohibitively expensive

$$z^{k+1} = z^k - (F'(x^k)^{-1} F(x^k) \quad \text{Where k=0, 1, 2. .} \quad (1.1)$$

Here, $F'(x^k)$ denotes the derivative or Jacobian matrix of F evaluated at x^k and $(F'(x^k))^{-1}$ is its inverse.

Statement of the problem

Numerical methods can be suitable for problems that are very difficult or impossible to solve analytically [13]. [10] developed extrapolated Newton's method in functions of one variable Perhaps, a more interesting and more useful application of root finding is to solve systems of nonlinear equations. Over the years, Newton-Raphson [12] iterative method has proved to be efficient method for solving system of non-linear equations. This study mainly deepened on the results of [1] and [10] in which extrapolated Newton-Raphsan method for solving the functions of one variable is extended to functions of two variables

Objective the study

- To describe the procedures and the techniques that can be followed in extrapolated Newton Raphson method for solving functions of two variables.
- To analyze the convergence of the proposed method.
- To explain the advantages of the proposed method over the existing methods

Preliminary consider the system of two equations and two unknowns:

$$f(x,y)=0$$

$$g(x,y)=0 \quad (4.1)$$

Let (x_k, y_k) be a suitable approximation to the root (ξ , η) of the system (4.1)

letΔx be an increment in x_k and Δy be an increment in y_k such that $(x_k+\Delta x, y_k+\Delta y)$ is the exact solution, that

$$f(x_k+\Delta x, y_k+\Delta y) \equiv 0$$

$$g(x_k+\Delta x, y_k+\Delta y) \equiv 0 \quad (4.2)$$

Expanding in Taylor series about the point (x_k, y_k), we get

$$f(x_k,y_k)+\left[\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right]f(x_k,y_k)+\frac{1}{2!}\left[\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right]^2 f(x_k,y_k)+\ldots=0$$

$$g(x_k,y_k)+\left[\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right]g(x_k,y_k)+\frac{1}{2!}\left[\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right]^2 g(x_k,y_k)+\ldots=0$$

(4.3)

Neglecting second and higher powers of Δx *and* Δy,we obtain

$$f(x_k,y_k)+\Delta x f_x(x_k,y_k)+\Delta y f(x_k,y_k)=0$$

$$g(x_k,y_k+\Delta x g_x(x\ ,y_k)+\Delta y g_y(x_k,y_k)=0 \quad (4.4)$$

Where suffixes with respect to x and y represent partial differentiation solving equation (4.4) for $\Delta x\ and\ \Delta y, we\ get$

$$\Delta x = -\frac{1}{D_k}\left[f(x_k, y_k) g_y(x_k, y_k) - g(x_k, y_k) f_y(x_k, y_k)\right]$$

$$\Delta y = \frac{1}{D_k}\left[g(x_{k,} y_k) f_x(x_k, y_k) - f(x_k, y_k) g_x(x_k, y_k)\right] \text{(4.5)}$$

Where: $D_k = f_x(x_k, y_k) g_y(x_k, y_k) - g_x(x_k y_k) f_y(x_k, y_k) \neq 0$

We obtain,

$$x_{k+1} = x_k + \Delta x \ and \ y_{k+1} = y_k + \Delta y$$

Writing the equation (4.4) in matrix form, we get

$$\begin{bmatrix} f_x(x_k, y_k) & f_y(x_k, y_k) \\ g_x(x_k, y_k) & g_y(x_k, y_k) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -\begin{bmatrix} f(x_k, y_k \\ g(x_k, y_k \end{bmatrix} \quad (4.6)$$

$$or;$$
$$J_k \Delta x = -F(x_k, y_k$$

where; $J_k = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{(x_k, y_k)}$, $\quad F = \begin{bmatrix} f \\ g \end{bmatrix}_{(x_k, y_k)}$

$$\Delta x = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

The solution of the system (4.6) is given by

$$\Delta x = J_k^{-1} F(x_k, y_k)$$

where; $J_k^{-1} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}^{-1}_{(x_k, y_{k)}} = \frac{1}{D_k}\begin{bmatrix} g_y & -f_y \\ -g_x & f_x \end{bmatrix}_{(x_k, y_k)}$

Therefore, we can write,

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = J_k^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix}$$

And;

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - J_k^{-1} \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix} \text{(4.7)} \quad \begin{matrix} where, \\ k = 0,1,2,\ldots \end{matrix}$$

Or;

$$x^{(k+1)} = x^{(k)} - J_k^{-1} F(x^{(k)}) \qquad (4.8)$$

where, $x^{(k)} = \left[x^{(k)}, y^{(k)}\right]^T$

And, $F(x^{(k)} = \left[f(x_k, y_k), g(x_k, y_k)\right]^T$

The method given by (4.8) is an extension of the Newton- Raphson method of equation $f(x) = 0$ to system of equations.

Extrapolated newton's method

Newton-Raphson method for solving systems of two non linear equations:

$\begin{matrix} f(x, y) = 0 \\ g(x, y) = 0 \end{matrix}$, in two variables $x\, and\, y$

In order to do this we must combine these two equations into a single equations form:

$F(z) = o$ (4.9)

where; $z = (x, y)$

Where F must give us a two component column vector and '0' (zero) is also a two component zero column vector. That is;

$$F(z) = \begin{bmatrix} f(z) \\ g(z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which is equivalent to the system of two equations $f(z) = 0$ and $g(z) = 0$

In order to do this we have to generalize the one variable Newton- Raphson method for solving the equation $f(x) = 0$ given by the sequence for x_n for x_0 the starting guess

Where; $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Or; $g(x_n) = x_{n+1}$

Where, $g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$

We can write equation (4.9) in the form $z = \phi(z^k)$ and generating sequence of approximation defined by:

$z^{k+1} = \phi(z^k)$ (4.10)

Where; $z^{k+1} = \begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix}^T$, Starting with z0 it is known that this sequence converge if $\left|\phi'(z^k)\right| < 1$ (4.11)

,for all open disc $D \in \Re^2$

Newton-Raphson method for solving (4.9) is given by:

$z^{k+1} = z^k - J^{-1}(z^k)F(z^k)$ (4.12)

where; $J^{-1}(z^k) = F'(z^k)$

And k=0, 1, 2, …

We extrapolate Newton's method,

$z^{k+1} = z^k - J^{-1}(z^k)F(z^k)$

$z^{k+1} = \partial_n z^{*k+1} + (1 - \partial_n)z^k$

Where; $z^{*k+1} = z^k - J^{-1}(z^k)F(z^k)$

$\Rightarrow z^{k+1} = z^k - \partial_n J^{-1}(z^k)F(z^k)$ (4.13)

By Fernando.T.G, in [3] we have the following relations:

where; $F'(z^k) \in R^2 \mathrm{X} R^2$

$F'(z^k) = J(z^k) = \nabla F^T(z^k)$ (4.14)

From equation (4.10) and equation (4.13), we have;

$$\phi(z^k) = z^k - \partial_n J^{-1}(z^k)F(x^k)$$

Since from (4.11), $\left|\phi'(z^k)\right| < 1$

$$\Rightarrow \left|\phi'(z^k)\right| = \left|1 - \partial_n \left(\frac{F(z^k)}{J(z^k)}\right)'\right| < 1$$

$$= \left|1 - \partial_n \left(\frac{F'(z^k)J(z^k) - J'(z^k)F(z^k)}{J^2(z^k)}\right)\right| < 1$$

$$= \left|1 - \partial_n \left(J^{-2}(z^k)[F'(z^k)J(z^k) - J'(z^k)F(z^k)]\right)\right| < 1$$

$$= \left|1 - \partial_n \left(J^{-1}(z^k)F'(z^k) - \frac{J'(z^k)F(z^k)}{J^2(z^k)}\right)\right| < 1$$

; By multiplying $\frac{F(z^k)}{F(z^k)}$ in bracket

$$= \left|1 - \partial_n \left(1 - \frac{J'(z^k)F(z^k)}{J(z^k)F(z^K)}\left(\frac{F(z^k)}{J(z^k)}\right)\right)\right| < 1$$

$$= \left|1 - \partial_n \left(1 - \frac{J'(z^k)F^2(z^k)}{J^2(z^k)F(z^k)}\right)\right| < 1$$ Because from (4.14) we have;

$$F'(z^k) = J(z^k)$$

$$= \left|1 - \partial_n + \partial n\left(\frac{J'(z^k)F^2(z^k)}{J^2(z^k)F(z^k)}\right)\right| < 1$$

Let $$\rho_n = \frac{J'(z^k)F^2(z^k)}{J^2(z^k)F(z^k)} \quad (4.15)$$

Where, (n=0, 1,2, . . .)

Or, by using equation (4.14), we obtain

$$\rho_n = \frac{F''(z^k)F^2(z^k)}{F'^2(z^k)F(z^k)} \quad (4.15)$$

where; $F'(z^k) = J(z^k) = \nabla F^T(z^k)$

$$\Rightarrow F''(z^k) = J'(z^k) = \nabla(\nabla F^T(z^k)) \quad (4.16)$$

Then, $\left|\phi'(z^k)\right| = \left|1 - \partial_n + \partial_n \rho_n\right| < 1$

Since the iteration method (4.10) is convergent under the condition of extrapolated Newton's method for functions of two variables will converge if,

$$\left|\phi'(z^k)\right| = \left|1 - \partial_n + \partial_n \rho_n\right| < 1$$

Let $\rho_n = \dfrac{J'(z^k)F^2(z^k)}{J^2(z^k)F(z^k)}$ (4.15)

Where, (n=0, 1,2, . . .)

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$$\rho_n = \frac{F''(z^k)F^2(z^k)}{F'^2(z^k)F(z^k)} \quad (4.15)$$

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$$\Rightarrow F''(z^k) = J'(z^k) = \nabla(\nabla F^T(z^k)) \quad (4.16)$$

Then, $\left|\phi'(z^k)\right| = \left|1 - \partial_n + \partial_n \rho_n\right| < 1$

Since the iteration method (4.10) is convergent under the condition of extrapolated Newton's method for functions of two variables will converge if,

$$\left|\phi'(z^k)\right| = \left|1 - \partial_n + \partial_n \rho_n\right| < 1$$

Let $\mu = \left|\phi'(z^k)\right|$

$$\Rightarrow \mu = \left|1-\partial_n+\partial_n\rho_n\right|<1 \quad ; \quad \forall z \subset D \in \Re^2 \text{ (4.17)}$$

From (4.17) we have;

$$\Rightarrow \left|1-\partial_n+\partial_n\rho_n\right|<1$$

$$\Rightarrow -1<\left(1-\partial_n+\partial_n\rho_n\right)<1$$

$$\Rightarrow -1<\left(1-\partial_n(1-\rho_n)\right)<1$$

$$\Rightarrow -2<\left(-\partial_n(1-\rho_n)\right)<0$$

$$\Rightarrow -2<-\partial_n(1-\rho_n) \text{ And } -\partial_n(1-\rho_n)<0$$

$$\Rightarrow 2>\partial_n(1-\rho_n) \text{ And } \partial_n(1-\rho_n)>0$$; by multiply '-ve' on both sides

$$\partial_n<\frac{2}{1-\rho_n} \,\&\, \partial_n>0$$

$$\Rightarrow \partial_n \in \left(0, \frac{2}{1-\rho_n}\right)$$

$$1-\rho_n<\frac{2}{\partial_n} \,\&\, \rho_n<1$$

Similarly;

$$\Rightarrow \rho_n>1-\frac{2}{\partial_n} \,\&\, \rho_n<1$$

$$\Rightarrow \rho_n \in \left(\left(1-\frac{2}{\partial_n}\right),1\right), \quad \forall \partial_n \in \left(0, \frac{2}{1-\rho_n}\right)$$

$$\Rightarrow \rho_n \in (-\infty,1)$$

Without loss of generality let consider the positive value of ρ_n that is $\rho_n \in (0,1)$.This is possible when we assume $J'(z_k)$ and $F(z_k)$ have the same sign.

Estimation of parameter

We need to find a positive real value of ∂_n for an each iteration, which minimizes μ of (4.17).

Since ρ_n of (4.15) is positive and real for all, we have in general;

$$a = 0 \le \rho_n \le \frac{J'(z^k)F^2(z^k)}{J^2(z^k)F(z^k)} = b \qquad (4.18)$$

Where; k=0, 1, 2,

The process of minimizing μ of (4.17) keeping in view of (4.18) with respect to ∂_n using the procedure give in young [16] gives the optimal choices for ∂_n as:

$$\partial_n(opt) = \frac{2}{2-(a+b)}$$

$$= \frac{2}{2-\rho_n} \qquad (4.19)$$

With the choice of computational parameter ∂_n, we have

$$\phi'(z^k) = 1 - \frac{2}{2-\rho_n} + \frac{2\rho_n}{2-\rho_n}$$

$$= \frac{(2-\rho_n)-2+2\rho_n}{2-\rho_n}$$

$$= \frac{\rho_n}{2-\rho_n} \qquad (4.20)$$

Numerical examples.

Consider the system of nonlinear equations.

$$f(x,y) = x^2 - y - 1 = 0$$
$$g(x,y) = y^2 - x = 0$$

The solution of the above system is obtained by using the Newton's method and Extrapolated Newton Raphson method taking initial approximation for x's and y's, error tolerance with 5x10^{-4} for both methods and the results are presented in Table4.1andTable4.2 respectively.

Table 1

K	$Z^{(K)}$	$Z^{(K+1)}$	$F(Z^{K})$
0	1	1.6667	-1
	1	1.3333	0
1	1.6667	1.5023	0.4444
	1.3333	1.23	0.1111
2	1.5023	1.4887	0.027

	1.23	1.2474	0.0107
3	1.4887	1.4903	-0.03117
	1.2474	1.2208	0.0073
4	1.4903	1.4902	0.0001
	1.2208	1.2207	0.00005

Table 2

K	$Z^{(K)}$	$Z^{(K+1)}$	$F(Z^{K})$
0	1	1.3333	-1
	1	1.25	0
1	1.3333	1.48760	-0.4723
	1.25	1.2257	0.2292
2	1.48760	1.4907	-0.0127
	1.2257	1.2204	0.0147
3	1.4907	1.4901	0.0017
	1.2204	1.2206	-0.0013

Discussion

The other thing is the interval of extrapolated parameter which controls the convergence of the method and optimal parameter is obtained by using the procedure of Young [16]. It is observed that the newly developed method is better in accuracy than the Newton-Raphson method as its order of convergence is third order while that of the later one is second order convergent. One practical example of systems of nonlinear equations is considered. The analysis of the results shows that the Newton's method takes 4 iterations to converge and Extrapolated Newton's method converges after three iterations as per the error of tolerance considered. As it shown in the analysis part and the example we considered the newly method is better than Newton-Raphson method for solving system of non-linear equations.

Methods	Number of iterations	Numerical solutions Z^{k+1}	**Numerical solutions**Z^{K}	**Absolute Errors** ($e=z^{k+1}-z^{k}$)
NM	0	1.6667	1	0.6667
		1.3333	1	0.3333
	1	1.5023	1.6667	0.1644
		1.23	1.3333	0.1033
	2	1.4887	1.5023	0.0136
		1.2474	1.23	0.0174
	3	1.4903	1.4887	0.0084
		1.2208	1.2474	0.0066
	4	1.4902	1.4903	0.0001
		1.2207	1.2208	0.0001
ENRM	0	1.3333	1	0.3333
		1.25	1	0.25
	1	1.48760	1.3333	0.1543
		1.2257	1.25	0.0243
	2	1.4907	1.48760	0.0031
		1.2204	1.2257	0.0053
	3	1.4907	1.4901	0.0006
		1.2204	1.2206	0.0002

Conclusion

The intention of this study is to introduced a new method and compare its result with that of Newton-Raphson method. Thus, from discussion of the results, it is easily observed that Extrapolated Newton's method is the efficient the method to solve systems of nonlinear equations with two variables. One practical example of systems of nonlinear equations is considered. Analysis of the results shows that the Newton's method takes 4 iterations to converge and Extrapolated Newton's method converges after three iterations as per the error of toleranceconsidered. As it shown in the analysis part and the example we considered the newly method is better than Newton-Raphson method for solving system of non-linear equations.

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