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Decomposition Of Balanced Complete Bipartite Graphs Into Cycles Of Two Different Length

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We decomposing the balanced complete bipartite graph $K_{2n,2n}$ into C_4's and C_{4n}'s. In particular, we find necessary and sufficient conditions for accomplishing this when $n \geq 2$, for $n \equiv 0(\mathrm{mod}2)$. As a consequence, we show that for nonnegative integers p and q, with $n \geq 2$, there exists a decomposition of the balanced complete bipartite graph $K_{2n,2n}$ into p copies of C_{2n} and q copies of C_4 if and only if $2np + 4q = 4n^2$, except when p is odd and n is even.

Keywords: Cycle, Complete bipartite graph, Graph Decomposition

1 Introduction

Unless stated otherwise all graphs considered here are finite, simple, and undirected. For the standard graph-theoretic the readers are referred to [1]. A cycle of length m is called m-cycle and it is denoted by C_m. Let K_m, I_m respectively denote a complete graph and an independent set on m vertices. $K_{m,n}$ denotes the complete bipartite graph with m and n vertices in the parts. A graph whose vertex set is partitioned into sets $V_1, \dots, V_m$ such that the edge set is $\bigcup_{i \neq j \in [m]} V_i \times V_j$ is a *complete m-partite graph*, denoted by $K_{n_1,\dots,n_m}$ when $|V_i| = n_i$ for all i. For any integer $\lambda > 0$, λG denotes the graph consisting of λ edge-disjoint copies of G. The complement of the graph G is denoted by $\overline{G}$. Let $(x_0 x_1 \dots x_{k-1} x_0)$ denote the cycle C_k with vertices $x_0, x_1, \dots, x_{k-1}$ and edges $x_0x_1, x_1x_2, \dots, x_{k-2}x_{k-1}, x_{k-1}x_0$. The *$\lambda$-multiplication* of G, denoted $G(\lambda)$, is the multigraph obtained from a graph G by replacing each edge with λ edges. For two graphs G and H, their *lexicographic product* or *wreath product* $G \otimes H$ has vertex set $V(G) \times V(H)$ with two vertices (g_1, h_1) and (g_2, h_2) adjacent whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. The complement of the graph G is denoted by $\overline{G}$.

By a *decomposition* of a graph G, we mean a list of edge-disjoint subgraphs of G whose union is G (ignoring isolated vertices). For a graph G, if $E(G)$ can be partitioned into $E_1, \dots, E_k$ such that the subgraph induced by E_i is H_i, for all i, $1 \le i \le k$, then we say that $H_1, \dots, H_k$ *decompose* G and we write $G = H_1 \oplus \dots \oplus H_k$, since $H_1, \dots, H_k$ are edge-disjoint subgraphs of G. For $1 \le i \le k$, if $H_i \cong H$, we say that G has a H-*decomposition*. A cycle passing through all the vertices of G is called *hamilton cycle* of G. An n-regular graph G is said to have a *Hamilton cycle decomposition* if its edge set can be partitioned into $n/2$ Hamilton cycles when n is even. If G has a decomposition into p copies of H_1 and q copies of H_2, then we say that G has a $\{pH_1, qH_2\}$-*decomposition*. If such a decomposition exists for all values of p and q satisfying trivial necessary conditions, then we say that G has a $\{H_1, H_2\}_{\{p,q\}}$-*decomposition* or G is *fully* $\{H_1, H_2\}$-*decomposable*.

Study of $\{H_1, H_2\}_{\{p,q\}}$-decomposition for graphs is not new. Chou et al. [2] proved that for a given triple (p, q, r) of nonnegative integers, G decompose into p copies of C_4, q copies of C_6, and r copies of C_8 such that $4p + 6q + 8r = |E(G)|$ in the following two cases: (a) $G = K_{m,n}$ with m and n both even at least 4, except $K_{4,4}$, (b) G is obtained from $K_{n,n}$ with n odd by deleting a perfect matching. Chou and Fu [3] proved that the existence of $\{C_4, C_{2t}\}_{\{p,q\}}$-decomposition of $K_{2u,2v}$, where $t/2 \le u, v < t$ when t even (resp., $(t+1)/2 \le u, v \le (3t-1)/2$ when t odd) implies such decomposition in $K_{2m,2n}$, where $m, n \ge t$ (resp., $m, n \ge (3t+1)/2$). Jeevadoss and Muthusamy [4] reduced the bounds in the sufficient conditions obtained by Chou and Fu [3] for the existence of $\{C_4, C_{2t}\}_{\{p,q\}}$-decomposition of $K_{2m,2n}$, when $t > 2$.

In this paper, we study the existence of $\{C_4, C_{4n}\}_{\{p,q\}}$-decomposition of $K_{2n,2n}$. In fact, we establish some necessary and sufficient conditions for the existence of $\{C_4, C_{4n}\}_{\{p,q\}}$-decomposition of $K_{2n,2n}$.

Let $K_{n,n}$ be the complete bipartite graph with bipartition (X, Y), where $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$. For $0 \le i \le n-1$, let $F_i(X, Y)$ denote the set $\{x_j y_{j+i} : j \in [n]\}$, where subscripts are taken modulo n. Clearly $F_i(X, Y)$ is a 1-factor of $K_{n,n}$, called the 1-factor of *distance* i. Also, $\bigcup_{i=0}^{n-1} F_i(X, Y) = K_{n,n}$. In a complete bipartite graph with bipartition (X, Y) with $|X| = |Y|$, an edge $x_i y_j$ is called an edge of *distance* $j - i$ if $i \le j$, or $n - (i - j)$, if $i > j$, from X to Y. (The same edge is said to be of *distance* $i - j$ if $i \ge j$ or $n - (i - j)$, if $i < j$, from Y to X.

Remark 1.1

i. For $n \in \mathbb{N}$, let $K_{2n,2n}$ have partite sets $X_1 \cup X_3$ and $X_2 \cup X_4$, where $X_r = \{x_1^r, \dots, x_n^r\}$. For $0 \le i \le n-1$, let $F_i(X_r, X_s) = \{x_j^r x_{j+i}^s : j \in [n]\}$, where arithmetic in subscripts is taken modulo n. Note that the union of the sets $F_i(X_r, X_s)$ over all i and $(r, s) \in \{(1,2), (2,3), (3,4), (4,1)\}$ decomposes $K_{2n,2n}$.
ii. For $k \in \{0, \dots, n-1\}$ and $i \in \{1,2,3,4\}$, the set $F_k(X_i, X_{i+1}) \cup F_{k+1}(X_i, X_{i+1})$ forms a 2-regular subgraph of $K_{2n,2n}$ consisting a cycle of length $2n$.
iii. For any positive integer n, the set $F_0(X_1, X_2) \cup F_0(X_2, X_3) \cup F_0(X_3, X_4) \cup F_1(X_4, X_1)$ forms a

Hamilton cycle of $K_{2n,2n}$.

iv. $F_k(X_i, X_j) = F_{n-k}(X_j, X_i)$, where $0 \le k \le n-1$.

v. For odd n, the set $F_{n-1}(X_1, X_2) \oplus F_0(X_2, X_3) \oplus F_{n-1}(X_3, X_4) \oplus F_0(X_4, X_1)$ forms a Hamilton cycle of $K_{2n,2n}$.

vi. For even n, the set $F_{n-1}(X_1, X_2) \oplus F_0(X_2, X_3) \oplus F_n(X_3, X_4) \oplus F_0(X_4, X_1)$ forms a Hamilton cycle of $K_{2n,2n}$.

vii. The edges of $F_{n-k}(X_1, X_2) \oplus F_k(X_2, X_3) \oplus F_{n-k}(X_3, X_4) \oplus F_k(X_4, X_1)$ forms a C_4 decomposition of $K_{2n,2n}$ where $0 \le k \le n-1$.

viii. The edges of $F_j(X_1, X_2) \oplus F_{j+1}(X_1, X_2)$, can be decomposed into P_3,s such that any two consecutive vertices x_r^1, x_{r+1}^1, $1 \le r \le n$ serve as end vertices in exactly one component in the P_3- decomposition. Similarly the edges of $F_k(X_1, X_4) \oplus F_{k+1}(X_1, X_4)$, can be decomposed into P_3,s such that any two consecutive vertices x_r^1, x_{r+1}^1, $1 \le r \le n$, serve as end vertices in exactly one component in the P_3- decomposition, thus $(x_r^1 x_a^2 x_b^4 x_{r+1}^1)$ forms a four cycle. Thus $F_j(X_1, X_2) \oplus F_{j+1}(X_1, X_2) \oplus F_k(X_1, X_4) \oplus F_{k+1}(X_1, X_4)$ can be decomposed into 4-cycles. Also $F_j(X_3, X_2) \oplus F_{j+1}(X_3, X_2) \oplus F_k(X_3, X_4) \oplus F_{k+1}(X_3, X_4)$ can be decomposed into 4-cycles, where $0 \le j, k \le n-1$.

2 $\{C_4, C_{4n}\}_{\{p,q\}}$-decomposition of $K_{2n,2n}$.

In this section we investigate the decompositions of $K_{2n,2n} - \alpha H$ into C_4, where αH denotes the α edge-disjoint Hamilton cycles of $K_{2n,2n}$.

Theorem 2.1 For odd $n \ge 3$, the graph $K_{2n,2n} - H$ can be decomposed into 4-cycles.

Proof. Without loss of generality, let

$$V(K_{2n,2n}) = (X_1 \cup X_3, X_2 \cup X_4),$$

$$E(K_{2n,2n} - H) = \bigcup_{k=0}^{n-1} (F_k(X_1, X_2) \oplus F_k(X_2, X_3) \oplus F_k(X_3, X_4) \oplus F_k(X_4, X_1)) \backslash H$$

where $H = F_{n-1}(X_1, X_2) \oplus F_0(X_2, X_3) \oplus F_{n-1}(X_3, X_4) \oplus F_0(X_4, X_1)$, is an Hamilton cycle of $K_{2n,2n}$.

$$K_{2n,2n} - H = \bigoplus_{k=0}^{\frac{n-3}{2}} \left[A_{2k,2k+1}(X_1, X_2) \oplus A'_{n-(2k+1),n-(2k+2)}(X_4, X_1)\right] \oplus \bigoplus_{k=0}^{\frac{n-3}{2}} \left[B_{n-(2k+1),n-(2k+2)}(X_2, X_3) \oplus B'_{2k,2k+1}(X_3, X_4)\right],$$

By Remark 1.1,

$$K_{2n,2n} - H = \bigoplus_{k=0}^{\frac{n-3}{2}} \left[A_{2k,2k+1}(X_1, X_2) \oplus A'_{2k+1,2k+2}(X_1, X_4)\right] \oplus \bigoplus_{k=0}^{\frac{n-3}{2}} \left[B_{2k+1,2k+2}(X_3, X_2) \oplus B'_{2k,2k+1}(X_3, X_4)\right],$$

where

$$A_{2k,2k+1}(X_1, X_2) = F_{2k}(X_1, X_2) \oplus F_{2k+1}(X_1, X_2)$$

$$A'_{2k+1,2k+2}(X_1,X_4)=F_{2k+1}(X_1,X_4)\oplus F_{2k+2}(X_1,X_4)$$
$$B_{2k+1,2k+2}(X_3,X_2)=F_{2k+1}(X_3,X_2)\oplus F_{2k+2}(X_3,X_2)$$
$$B'_{2k,2k+1}(X_3,X_4)=F_{2k}(X_3,X_4)\oplus F_{2k+1}(X_3,X_4).$$

By Remark 1.1, $A_{2k,2k+1}(X_1,X_2)\oplus A'_{2k+1,2k+2}(X_1,X_4)$ can be decomposed into 4-cycles, similarly $B_{2k+1,2k+2}(X_3,X_2)\oplus B'_{2k,2k+1}(X_3,X_4)$ can be decomposed in 4-cycles, we obtain the proof.

Theorem 2.2 For odd $n\geq 3$, odd $\alpha, 1\leq\alpha\leq n$, the graph $K_{2n,2n}-\alpha H$ can be decomposed into 4-cycles.

Proof. Without loss of generality, let

$$V(K_{2n,2n})=(X_1\cup X_3, X_2\cup X_4)\quad \text{where} X_i=\{x_1^i,x_2^i,\cdots,x_n^i\},$$
$$E(K_{2n,2n}-\alpha H)=\bigcup_{k=0}^{n-1}[F_k(X_1,X_2)\oplus F_k(X_2,X_3)\oplus F_k(X_3,X_4)\oplus F_k(X_4,X_1)]\setminus\bigcup_{p=0}^{\alpha-1}H_p,$$

where $H_p=F_{n+1-p}(X_1,X_2)\oplus F_p(X_2,X_3)\oplus F_{n-p}(X_3,X_4)\oplus F_p(X_4,X_1)$, for $0\leq p\leq\alpha-1$ are edge disjoint Hamilton cycles of $K_{2n,2n}$.

$$K_{2n,2n}-\alpha H=\bigoplus_{k=0}^{\frac{n-\alpha-2}{2}}\left[A_{2k+2,2k+3}(X_1,X_2)\oplus A'_{n-(2k+1),n-(2k+2)}(X_4,X_1)\right]\oplus$$
$$\bigoplus_{k=0}^{\frac{n-\alpha-2}{2}}\left[B_{n-(2k+1),n-(2k+2)}(X_2,X_3)\oplus B'_{2k+1,2k+2}(X_3,X_4)\right].$$

By Remark 1.1,

$$K_{2n,2n}-\alpha H=\bigoplus_{k=0}^{\frac{n-\alpha-2}{2}}\left[A_{2k+2,2k+3}(X_1,X_2)\oplus A'_{2k+1,2k+2}(X_1,X_4)\right]\oplus$$
$$\bigoplus_{k=0}^{\frac{n-\alpha-2}{2}}\left[B_{2k+1,2k+2}(X_3,X_2)\oplus B'_{2k+1,2k+2}(X_3,X_4)\right],$$

where

$$A_{2k+2,2k+3}(X_1,X_2)=F_{2k+2}(X_1,X_2)\oplus F_{2k+3}(X_1,X_2)$$
$$A'_{2k+1,2k+2}(X_1,X_4)=F_{2k+1}(X_1,X_4)\oplus F_{2k+2}(X_1,X_4)$$
$$B_{2k+1,2k+2}(X_3,X_2)=F_{2k+1}(X_3,X_2)\oplus F_{2k+2}(X_3,X_2)$$
$$B'_{2k+1,2k+2}(X_3,X_4)=F_{2k+1}(X_3,X_4)\oplus F_{2k+2}(X_3,X_4).$$

By Remark 1.1, $A_{2k+2,2k+3}(X_1,X_2)\oplus A'_{2k+1,2k+2}(X_1,X_4)$ can be decomposed into 4-cycles, similarly $B_{2k+1,2k+2}(X_3,X_2)\oplus B'_{2k+1,2k+2}(X_3,X_4)$ can be decomposed in 4-cycles, we obtain the proof.

Theorem 2.3 For odd $n\geq 3$, even $\alpha, 1\leq\alpha\leq n$, the graph $K_{2n,2n}-\alpha H$ can be decomposed into 4-cycles.

Proof. Without loss of generality,

Let $V(K_{2n,2n})=(X_1\cup X_3, X_2\cup X_4)\text{where} X_i=\{x_1^i,x_2^i,\cdots,x_n^i\}$,

For $\alpha<n-1$,

$$E(K_{2n,2n}-\alpha H)=\bigcup_{k=0}^{n-1}[F_k(X_1,X_2)\oplus F_k(X_2,X_3)\oplus F_k(X_3,X_4)\oplus F_k(X_4,X_1)]\setminus\bigcup_{p=0}^{\alpha-1}H_p,$$

where $H_p=F_{n-1+p}(X_1,X_2)\oplus F_{n-1-p}(X_2,X_3)\oplus F_{n-1-p}(X_3,X_4)\oplus F_{n-1+p}(X_4,X_1)$, for $0\le p\le\alpha-1$ are edge disjoint Hamilton cycles of $K_{2n,2n}$.

$$K_{2n,2n}-\alpha H=\bigoplus_{k=0}^{\frac{n-\alpha-3}{2}}\left[A_{n-(2k+3),n-(2k+4)}(X_1,X_2)\oplus A'_{2k+3,2k+4}(X_1,X_4)\right]\oplus$$
$$\bigoplus_{i=0}^{\frac{n-\alpha-5}{2}}\left[B_{n-(2k+3),n-(2k+4)}(X_3,X_2)\oplus B'_{2k+3,2k+4}(X_3,X_4)\right]\oplus Y\oplus Z,$$

For $\alpha=n-1$,

$$E(K_{2n,2n}-\alpha H)=\bigcup_{k=0}^{n-1}[F_k(X_1,X_2)\oplus F_k(X_2,X_3)\oplus F_k(X_3,X_4)\oplus F_k(X_4,X_1)]\setminus$$
$$\left[\bigcup_{p=0,p\neq\alpha-2}^{\alpha-1}H_p\oplus W\right],$$

where $H_p=F_{n-1+p}(X_1,X_2)\oplus F_{n-1-p}(X_2,X_3)\oplus F_{n-1-p}(X_3,X_4)\oplus F_{n-1+p}(X_4,X_1)$, for $0\le p(\neq\alpha-2)\le\alpha-1$ are edge disjoint Hamilton cycles of $K_{2n,2n}$.

$$K_{2n,2n}-\alpha H=Z$$

where,

$$A_{n-(2k+3),n-(2k+4)}(X_1,X_2)=F_{n-(2k+3)}(X_1,X_2)\oplus F_{n-(2k+4)}(X_1,X_2)$$
$$A'_{2k+3,2k+4}(X_1,X_4)=F_{2k+3}(X_1,X_4)\oplus F_{2k+4}(X_1,X_4)$$
$$B_{n-(2k+3),n-(2k+4)}(X_3,X_2)=F_{n-(2k+3)}(X_3,X_2)\oplus F_{n-(2k+4)}(X_3,X_2)$$
$$B'_{2k+3,2k+4}(X_3,X_4)=F_{2k+3}(X_3,X_4)\oplus F_{2k+4}(X_3,X_4).$$
$$Y=B_{0,n-1}(X_3,X_2)\oplus B'_{0,1}(X_3,X_4)$$
$$W=F_{n-4}(X_1,X_2)\oplus F_0(X_2,X_3)\oplus F_0(X_3,X_4)\oplus F_{n-4}(X_4,X_1)$$
$$Z=F_{n-2}(X_1,X_2)\oplus F_2(X_2,X_3)\oplus F_2(X_3,X_4)\oplus F_{n-2}(X_4,X_1)$$

By Remark 1.1, $A_{n-(2k+3),n-(2k+4)}(X_1,X_2)\oplus A'_{2k+3,2k+4}(X_1,X_4)$ can be decomposed into 4-cycles, similarly $B_{n-(2k+3),n-(2k+4)}(X_3,X_2)\oplus B'_{2k+3,2k+4}(X_3,X_4)$, Y and Z can be decomposed in 4-cycles, W is an Hamilton cycle , we obtain the proof.

Theorem 2.4 For even $n\ge 4$, even $\alpha, 1\le\alpha\le n$, the graph $K_{2n,2n}-\alpha H$ can be decomposed into 4-cycles.

Proof. Without loss of generality, Let

$$V(K_{2n,2n})=(X_1\cup X_3,X_2\cup X_4),\text{where } X_i=\{x_1^i,x_2^i,\cdots,x_n^i\},$$
$$E(K_{2n,2n}-\alpha H)=\bigcup_{k=0}^{n-1}[F_k(X_1,X_2)\oplus F_k(X_2,X_3)\oplus F_k(X_3,X_4)\oplus F_k(X_4,X_1)]\setminus\bigcup_{p=0}^{\alpha-1}H_p,$$

where $H_p=F_{n+1-p}(X_1,X_2)\oplus F_p(X_2,X_3)\oplus F_{n-p}(X_3,X_4)\oplus F_p(X_4,X_1)$, for $0\le p\le\alpha-1$ are edge disjoint Hamilton cycles of $K_{2n,2n}$.

$$K_{2n,2n} - \alpha H = \bigoplus_{k=0}^{\frac{n-\alpha-2}{2}} \left[A_{2k+2,2k+3}(X_1, X_2) \oplus A'_{2k+1,2k+2}(X_1, X_4)\right] \oplus \bigoplus_{k=0}^{\frac{n-\alpha-2}{2}} \left[B_{2k+1,2k+2}(X_3, X_2) \oplus B'_{2k+1,2k+2}(X_3, X_4)\right],$$

where

$$A_{2k+2,2k+3}(X_1, X_2) = F_{2k+2}(X_1, X_2) \oplus F_{2k+3}(X_1, X_2)$$
$$A'_{2k+1,2k+2}(X_1, X_4) = F_{2k+1}(X_1, X_4) \oplus F_{2k+2}(X_1, X_4)$$
$$B_{2k+1,2k+2}(X_3, X_2) = F_{2k+1}(X_3, X_2) \oplus F_{2k+2}(X_3, X_2)$$
$$B'_{2k+1,2k+2}(X_3, X_4) = F_{2k+1}(X_3, X_4) \oplus F_{2k+2}(X_3, X_4).$$

By Remark 1, $A_{2k+2,2k+3}(X_1, X_2) \oplus A'_{2k+1,2k+2}(X_1, X_4)$ can be decomposed into 4-cycles, similarly $B_{2k+1,2k+2}(X_3, X_2) \oplus B'_{2k+1,2k+2}(X_3, X_4)$ can be decomposed in 4-cycles, we obtain the proof.

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