Nat. Volatiles & Essent. Oils, 2021; 8(5): 3436 - 3445

Solving A Class Of Fredhlom Integral Inclusions Via Fixed Point Theorems In The Turf Of $\mathbb{D}_{\mathcal{C}}$-WD Mappings

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Abstract. In this manuscript, we define $\mathbb{D}_{\mathcal{C}}$-weakly demicompact mappings; we discuss the existence of coincidence points in the turf of a restricted class of $\mathbb{D}_{\mathcal{C}}$-weakly demicompact mappings defined on cone metric spaces; as a consequence of the result we authenticate the existence of common fixed points. Finally, we present an application to our core result.

Mathematics subject Classification. 47H10, 54H25.

Keywords. Common fixed point, Coincidence point, $\mathcal{H}$-cone metric space, $\mathbb{D}_{\mathcal{C}}$-weakly demicompact, Integral inclusions.

1 Introduction

The notion of cone metric spaces is defined and discussed by Huang [9]; later, Abbas and Jungck[1] derived certain significant fixed point results, without using continuity. Azam, Kumar and Wang[4, 13, 18] are some others who studied and extended the theory further in this context. Wardowski[17] introduced the idea of $\mathcal{H}$-cone metric to estimate the distance among two sets in a cone metric space and proved a Nadler's type of fixed point theorem.

In 2013, Arshad and Ahmad[3] studied the concept of fixed points of multivalued mappings without normal cone by modifying the definition of Wardowski. The equivalence between the theorems of Arshad, Ahmad and Nadler is proved by Huang et al.[10]. In 2002, Branciari[5] introduced a integral type contractive condition for mappings on metric space and proved certain fixed point results. Khojastech et al.[12] distended the theory in the field of cone metric spaces with normal cone. The concepts like commutativity, compatibility, weakly compatibility and R-weakly commutativity are defined and discussed (see [7, 11, 14, 15]).

In this paper, we define a hybrid pair of mapping namely $\mathbb{D}_{\mathcal{C}}$-weakly demicompact mapping. As sequel we prove that any $\mathbb{D}_{\mathcal{C}}$-weakly demicompact mapping which satisfies a specified integral type of contractive condition has a coincidence point; in follow we authenticate the existence of a common fixed point for a pair of $\mathbb{D}_{\mathcal{C}}$-weakly demicompact mappings. Finally, we present an application of our core result.

2 Preliminaries

Let E denotes the real Banach space and with zero element θ. Let $P \in \mathcal{P}(E) - \{\theta\}$, then P is said to be a cone if

(C1) $k\mu + l\nu \in P \; \forall \; \mu, \nu \in P$ and $k, l \in [0, \infty)$;

(C2) θ is the only member in P which has additive inverse in P.

Let $\preceq$ be a partial order in P defined by $\mu \preceq \nu$ if and only if $\nu - \mu \in P$. $\mu \prec \nu$ indicates $\mu \preceq \nu$ but $\mu \neq \nu$. $\mu \prec\prec \nu$ indicates $\nu - \mu$ belongs to interior of μ. μ is said to be normal if there exist $k \geq 0$ such that $||\mu|| \leq k||\nu|| \; \forall \; \mu, \nu \in E$ with $\theta \preceq \mu \preceq \nu$; such a least positive number is said to be a normal constant.

Definition 2.1 *[9] A function $\sigma: S^2 \to E$, is called as a cone metric if it satisfies:*

(CM1) $\theta \preceq \sigma(p,q) \; \forall \; p, q \in S$;

(CM2) $\sigma(p,q) = \theta$ iff $p = q$;

(CM3) $\sigma(p,q) = \sigma(q,p) \; \forall \; p, q \in S$;

(CM4) $\sigma(p,q) \preceq \sigma(p,r) + \sigma(r,q) \; \forall \; p, q, r \in S$.

In order to avoid ambiguity, we denote a cone metric space by S_c.

Definition 2.2 *[7] Let $A \neq \emptyset$ and $B \neq \emptyset$ be bounded closed subsets of a metric space S. Then Hausdorff metric is given by*

$$\mathcal{H}(A,B) = \max\{\sup_{p \in A} \inf_{q \in B} \sigma(p,q), \sup_{q \in B} \inf_{p \in A} \sigma(p,q)\}.$$

Definition 2.3 *[3] Let $\mathcal{C}(S) = \{A \subseteq S / A \text{ is closed}\}$. A map $\mathcal{H}: \mathcal{C}(S)^2 \to E$ is said to be $\mathcal{H}$-cone metric on $\mathcal{C}(S)$ if it satisfies*

(HC1) $\theta \preceq \mathcal{H}(A,B) \; \forall \; A, B \in \mathcal{C}(S)$;

(HC2) $\mathcal{H}(A,B) = \theta$ iff $A = B$;

(HC3) $\mathcal{H}(A,B) = \mathcal{H}(B,A) \; \forall \; A, B \in \mathcal{C}(S)$;

(HC4) $\mathcal{H}(A,B) \preceq \mathcal{H}(A,C) + \mathcal{H}(C,B) \; \forall \; A, B, C \in \mathcal{C}(S)$;

(HC5) If $A, B \in \mathcal{C}(S)$, $\theta \prec \epsilon \in E$ with $\mathcal{H}(A,B) \prec \epsilon$, then for each $a \in A$ there exists $b \in B$ such that $\sigma(a,b) \prec \epsilon$.

Let $v_1, v_2 \in E$ and $v_1 \prec v_2$. Define

$$[v_1, v_2] = \{v \in E: c = \lambda v_2 + (1-\lambda)v_1, \; for \; some \; \lambda \in [0,1]\}$$
$$[v_1, v_2) = \{v \in E: c = \lambda v_2 + (1-\lambda)v_1, \; for \; some \; \lambda \in [0,1)\}$$

The set $\{v_1 = u_0, u_1, \cdots, u_n = v_2\}$ is said to be a partition for $[v_1, v_2]$ iff the sets $\{[u_{i-1}, u_i)\}_{i=1}^{n}$ are pairwise disjoint and $[v_1, v_2] = \{\bigcup_{i=1}^{n} [u_{i-1}, u_i)\} \cup \{v_2\}$.

Let Q be a partition and $\phi: [v_1, v_2] \to P$ be an increasing function. Then

$$L_n = \sum_{i=0}^{n-1} \phi(u_i)||u_i - u_{i+1}|| \quad and \quad U_n = \sum_{i=0}^{n-1} \phi(u_{i+1})||u_i - u_{i+1}||$$

are called as cone lower sum and cone upper sum respectively. (see [12])

Definition 2.4 *[12] An increasing function $\phi: [v_1, v_2] \to P$ is said to be an integrable function simply, cone integrable function if*

$$\lim_{n \to \infty} L_n = U = \lim_{n \to \infty} U_n$$

where U must be unique. The common value of U is denoted by $\int_{v_1}^{v_2} \phi d_P$

Definition 2.5 *[11] Let $\mathbb{D}: S \to \mathfrak{P}(S)$ and $\mathcal{C}: S \to S$. A point $p \in S$ is said to be*

(i) a coincidence point of $\mathbb{D}$ and $\mathcal{C}$, if $\mathcal{C}p \in \mathbb{D}p$.
(ii) a common fixed point of $\mathbb{D}$ and $\mathcal{C}$, if $p = \mathcal{C}p \in \mathbb{D}p$.

Definition 2.6 *[2, 6, 16] Let $\mathcal{C}: S \to S$ and $\mathbb{D}: S \to \mathcal{P}(S)$.*

(i) The self mapping $\mathcal{C}$ is called as sequentially convergent if every sequence $\{p_n\} \in S$, $\{\mathcal{C}p_n\}$ is convergent implies $\{p_n\}$ is convergent.
(ii) The multivalued mapping $\mathbb{D}$ is called as weakly demicompact if every sequence $\{p_n\} \in S$ such that $p_{n+1} \in \mathbb{D}p_n$ and $\lim_{n\to\infty} \sigma(p_n, p_{n+1}) = \theta$, having a convergent subsequence $\{p_{n_k}\}_{k\in\mathbb{N}}$.
(iii) The pair $(\mathbb{D}, \mathcal{C})$ is called as weakly compatible if they commute at their coincidence point.

Definition 2.7 *[8] A sequence $\{p_n\}$ in S is called as asymptotically regular if $\lim_{n\to\infty} \sigma(p_{n+1}, p_n) = \theta$.*

3 Main results

Throughout this section, let ϕ be a nonvanishing self map on P which is subadditive cone integrable for any $[\mu, \nu] \subset P$ with $\theta \prec\prec \int_\theta^\epsilon \phi d_P$, $\forall\ \theta \prec \epsilon$.

Definition 3.1 *Let $\mathcal{A} = \mathcal{P}(S) - \emptyset$. A pair of mapping $\mathbb{D}: S \to \mathcal{A}$ and $\mathcal{C}: S \to S$ is called as $\mathbb{D}_{\mathcal{C}}$-weakly demicompact if for every asymptotically regular sequence $\{\mathcal{C}p_n\}$ in S with $\mathcal{C}p_{n+1} \in \mathbb{D}p_n$, there exists a convergent subsequence.*

Example 3.2 *Let* $E = \{\mu: [0,1] \to \mathbb{R} \mid \mu \text{ is continuous}\}$ and $P = \{\mu \mid 0 \le \mu(t)\ \forall\ t \in [0,1]\}$. *Let $\sigma: [0,1)^2 \to E$ be a function defined by $\sigma(p, q) = |p - q|e^t$. Then by considering E as a real Banach space with the normal cone P, σ is a cone metric on $[0,1)$. Let $\mathcal{H}: \mathfrak{C}([0,1))^2 \to E$ be a function defined as*

$$\mathcal{H}(\mathbb{D}p, \mathbb{D}q) = \mathcal{H}_u(\mathbb{D}p, \mathbb{D}q)e^t,$$

where $\mathcal{H}_u$ is the standard Hausdorff metric.

Let $\mathcal{C}: [0,1) \to [0,1)$ and $\mathbb{D}: [0,1) \to \mathcal{C}([0,1))$ be defined as $\mathcal{C}(p) = \sin p$ and $\mathbb{D}(p) = [0, \sin p]$. Let $p_0 \in [0,1)$. Choose a point p_1 such that

$$\mathcal{C}(p_1) = \sin p_1 \in [0, \sin p_0] = \mathbb{D}(p_0).$$

which implies that $\mathcal{C}(p_1) \le \sin p_0$. Again choose a point p_2 such that

$$\mathcal{C}(p_2) = \sin p_2 \in [0, \sin p_1] = \mathbb{D}(p_1).$$

It follows that $\mathcal{C}(p_2) \le \mathcal{C}(p_1)$. Proceeding like this we get a monotonically decreasing sequence $\cdots \le \mathcal{C}(p_n) \le \mathcal{C}(p_{n-1}) \le \cdots \le \mathcal{C}(p_2) \le \mathcal{C}(p_1)$ which is bounded below by 0 and hence $\lim_{n\to\infty} |\mathcal{C}(p_n)| = 0$. The sequence of the form $\{\mathcal{C}p_n : \mathcal{C}p_{n+1} \in \mathbb{D}p_n\}$ is convergent and $\lim_{n\to\infty} \sigma(\mathcal{C}p_n, \mathcal{C}p_{n+1}) = \theta$. Hence the pair of mappings $(\mathbb{D}, \mathcal{C})$ is $\mathbb{D}_{\mathcal{C}}$-weakly demicompact.

Let Ψ be the class of all continuous self mappings on a cone P that satisfies the following property

a) Each $\psi \in \Psi$ is subadditive and sequentially convergent;
b) $p \preceq q \Rightarrow \psi(p) \preceq \psi(q)\ \forall\ \psi \in \Psi$;
c) $\psi(p) = \theta$ iff $p = \theta\ \forall\ \psi \in \Psi$;
d) $\psi(kp) = k\psi(p)$ for some $k > 0\ \forall\ \psi \in \Psi$.

Theorem 3.3 *Let $(\mathbb{D}, \mathcal{C})$ be a $\mathbb{D}_{\mathcal{C}}$-weakly demicompact mapping such that $\mathcal{C}$ is onto and $\mathbb{D}(S) \subseteq \mathcal{C}(S)$. Let $\mathcal{H}$ be a $\mathcal{H}$-cone metric on $\mathcal{C}(S)$ such that for some $\lambda \in (0,1)$ and $\psi \in \Psi$,*

$$\psi\left(\int_\theta^{\mathcal{H}(\mathbb{D}p, \mathbb{D}q)} \phi d_P\right) \preceq \lambda \psi\left(\int_\theta^{\sigma(\mathcal{C}q, r)} \phi d_P\right) \forall\ p, q \in S \text{ and } r \in \mathbb{D}p, \quad (1)$$

then there exists $p \in S$ such that $\mathcal{C}p \in \mathbb{D}p$.

Proof. Let $p_0 \in S$ and $q_0 = \mathcal{C}p_0$. Since $\mathcal{C}$ is onto, there exists $p_1 \in S$ such that $q_1 = \mathcal{C}p_1 \in \mathbb{D}p_0$. Suppose $q_1 = q_0$, then $\mathcal{C}p_0 \in \mathbb{D}p_0$ as desired.

If $q_1 \neq q_0$, then $\theta \prec \sigma(q_1, q_0)$. Suppose $\epsilon = \mathcal{H}(\mathbb{D}p_0, \mathbb{D}p_1) + \lambda\sigma(q_0, q_1)$ where $\lambda > 0$, then we have $\mathcal{H}(\mathbb{D}p_0, \mathbb{D}p_1) \prec \epsilon$. But by (HC5), there exist $p_2 \in S$ such that $q_2 = \mathcal{C}p_2 \in \mathbb{D}p_1$, $\sigma(q_1, q_2) \prec \epsilon$. Thus we get that

$$\int_{\theta}^{\sigma(q_1,q_2)} \phi d_P \preceq \int_{\theta}^{\mathcal{H}(\mathbb{D}p_0,\mathbb{D}p_1)+\lambda\sigma(y_0,y_1)} \phi d_P$$
$$\preceq \int_{\theta}^{\mathcal{H}(\mathbb{D}p_0,\mathbb{D}p_1)} \phi d_P + \int_{\theta}^{\lambda\sigma(q_0,q_1)} \phi d_P$$

Since ψ is an increasing mapping, we have

$$\psi\left(\int_{\theta}^{\sigma(q_1,q_2)} \phi d_P\right) \preceq \psi\left(\int_{\theta}^{\mathcal{H}(\mathbb{D}p_0,\mathbb{D}p_1)} \phi d_P + \int_{\theta}^{\lambda\sigma(q_0,q_1)} \phi d_P\right)$$
$$\preceq \psi\left(\int_{\theta}^{\mathcal{H}(\mathbb{D}p_0,\mathbb{D}p_1)} \phi d_P\right) + \psi\left(\int_{\theta}^{\lambda\sigma(q_0,q_1)} \phi d_P\right)$$
$$\preceq \lambda\psi\left(\int_{\theta}^{\sigma(\mathcal{C}p_1,\mathcal{C}p_1)} \phi d_P\right) + \psi\left(\int_{\theta}^{\lambda\sigma(q_0,q_1)} \phi d_P\right)$$
$$\psi\left(\int_{\theta}^{\sigma(q_1,q_2)} \phi d_P\right) \preceq \psi\left(\int_{\theta}^{\lambda\sigma(q_0,q_1)} \phi d_P\right).$$

Now suppose $\epsilon = \mathcal{H}(\mathbb{D}p_1, \mathbb{D}p_2) + \lambda^2\sigma(q_0, q_1)$, then we have $\mathcal{H}(\mathbb{D}p_1, \mathbb{D}p_2) \prec \epsilon$. Again by (HC5), there exists $p_3 \in S$ so that $q_3 = \mathcal{C}p_3 \in \mathbb{D}p_2$, $\sigma(q_2, q_3) \prec \epsilon$. Analogous to the above argument it can be derived that

$$\psi\left(\int_{\theta}^{\sigma(q_2,q_3)} \phi d_P\right) \preceq \psi\left(\int_{\theta}^{\lambda^2\sigma(q_0,q_1)} \phi d_P\right)$$

Proceeding as above, we get $q_{n+1} = \mathcal{C}p_{n+1} \in \mathbb{D}p_n$ for some $p_{n+1} \in S$ and

$$\psi\left(\int_{\theta}^{\sigma(q_n,q_{n+1})} \phi d_P\right) \preceq \psi\left(\int_{\theta}^{\lambda^n\sigma(q_0,q_1)} \phi d_P\right)$$

Letting $n \to \infty$ on both sides, we get

$$\lim_{n\to\infty} \psi\left(\int_{\theta}^{\sigma(q_n,q_{n+1})} \phi d_P\right) \preceq \lim_{n\to\infty} \psi\left(\int_{\theta}^{\lambda^n\sigma(q_0,q_1)} \phi d_P\right)$$

As $\lambda \in (0,1)$, we have $\lim_{n\to\infty} \int_{\theta}^{\lambda^n\sigma(q_0,q_1)} \phi d_P = \theta$. Also since ψ is continuous, it follows that $\lim_{n\to\infty} \psi\left(\int_{\theta}^{\lambda^n\sigma(q_0,q_1)} \phi d_P\right) = \theta$ and therefore

$$\lim_{n\to\infty}\psi\left(\int_{\theta}^{\sigma(q_n,q_{n+1})}\phi d_P\right)=\theta.$$

But $\lim_{n\to\infty}\int_{\theta}^{\sigma(q_n,q_{n+1})}\phi d_P=\theta$, as ψ is sequentially convergent. Thus we have $\lim_{n\to\infty}\sigma(q_n,q_{n+1})=\theta$. Now since the pair $(\mathbb{D},\mathcal{C})$ are $\mathbb{D}_{\mathcal{C}}$-weakly demicompact, there exist a convergent subsequence $\{q_{n_k}\}$ of $\{q_n\}$. If we let $\lim_{k\to\infty}q_{n_k}=u$, then there exist a point $r\in S$ such that $\mathcal{C}r=u$.

Finally, we claim that r is the required coincidence point. Now suppose $\epsilon=\mathcal{H}(\mathbb{D}p_{n_k-1},\mathbb{D}r)+\lambda^{n_k}\sigma(q_0,q_1)$, then $\mathcal{H}(\mathbb{D}p_{n_k-1},\mathbb{D}r)\prec\epsilon$. By (HC5), there exists $r_{n_k}\in\mathbb{D}r$ such that

$$\sigma(\mathcal{C}p_{n_k},r_{n_k})\prec\mathcal{H}(\mathbb{D}p_{n_k-1},\mathbb{D}r)+\lambda^{n_k}\sigma(q_0,q_1).$$

Thus we have

$$\int_{\theta}^{\sigma(\mathcal{C}p_{n_k},r_{n_k})}\phi d_P\preceq\int_{\theta}^{\mathcal{H}(\mathbb{D}p_{n_k-1},\mathbb{D}r)+\lambda^{n_k}\sigma(q_0,q_1)}\phi d_P$$

$$\preceq\int_{\theta}^{\mathcal{H}(\mathbb{D}p_{n_k-1},\mathbb{D}r)}\phi d_P+\int_{\theta}^{\lambda^{n_k}\sigma(q_0,q_1)}\phi d_P$$

But since ψ is increasing and subadditive, we get

$$\psi\left(\int_{\theta}^{\sigma(\mathcal{C}p_{n_k},r_{n_k})}\phi d_P\right)\preceq\psi\left(\int_{\theta}^{\mathcal{H}(\mathbb{D}p_{n_k-1},\mathbb{D}r)}\phi d_P\right)+\psi\left(\int_{\theta}^{\lambda^{n_k}\sigma(q_0,q_1)}\phi d_P\right).$$

Then by (1), it follows that

$$\psi\left(\int_{\theta}^{\sigma(\mathcal{C}p_{n_k},r_{n_k})}\phi d_P\right)\preceq\lambda\psi\left(\int_{\theta}^{\sigma(\mathcal{C}p_{n_k},\mathcal{C}r)}\phi d_P\right)+\psi\left(\int_{\theta}^{\lambda^{n_k}\sigma(q_0,q_1)}\phi d_P\right).$$

By taking limit $k\to\infty$ on both sides we get

$$\lim_{k\to\infty}\psi\left(\int_{\theta}^{\sigma(\mathcal{C}p_{n_k},r_{n_k})}\phi d_P\right)\preceq\lambda\lim_{k\to\infty}\psi\left(\int_{\theta}^{\sigma(\mathcal{C}r,\mathcal{C}p_{n_k})}\phi d_P\right)+\lim_{k\to\infty}\psi\left(\int_{\theta}^{\lambda^{n_k}\sigma(q_0,q_1)}\phi d_P\right).$$

Since ψ is continuous, $\lambda\in(0,1)$, and $\lim_{k\to\infty}\sigma(\mathcal{C}r,\mathcal{C}p_{n_k})=\theta$, we have

$$\lim_{k\to\infty}\psi\left(\int_{\theta}^{\sigma(\mathcal{C}r,\mathcal{C}p_{n_k})}\phi d_P\right)=\lim_{k\to\infty}\psi\left(\int_{\theta}^{\lambda^{n_k}\sigma(q_0,q_1)}\phi d_P\right)=\theta.$$

Therefore, $\lim_{k\to\infty}\psi\left(\int_{\theta}^{\sigma(\mathcal{C}p_{n_k},r_{n_k})}\phi d_P\right)=\theta$, and as ψ is sequentially convergent,

$$\lim_{k\to\infty}\int_{\theta}^{\sigma(\mathcal{C}p_{n_k},r_{n_k})}\phi d_P=\theta.$$

Thus it follows that $\lim_{n\to\infty}r_{n_k}=p$. Now since $\{r_{n_k}\}$ is a sequence in $\mathbb{D}r$ and $\mathbb{D}r$ is closed, we have $u=$

$\mathcal{C}r \in \mathbb{D}r$ as desired.

Example 3.4 *Let* $E = \{\mu: [0,1] \to \mathbb{R} | \mu \text{ is continuous}\}$ *and* $P = \{\mu(t) | 0 \leq \mu(t) \forall\, t \in [0,1]\}$. *Let* $\sigma: (0,1)^2 \to E$ *be a function defined by* $\sigma(p, q) = |p - q|e^t$. *Then by considering* E *as a real Banach space with the normal cone* P, *it is easy to see that* σ *is a cone metric on* $(0,1)$. *Let* $\mathcal{H}: \mathfrak{C}((0,1))^2 \to E$ *be a function defined as*

$$\mathcal{H}(\mathbb{D}p, \mathbb{D}q) = \mathcal{H}_u(\mathbb{D}p, \mathbb{D}q)e^t,$$

where $\mathcal{H}_u$ is the standard Hausdorff metric.

Let $\mathbb{D}: (0,1) \to \mathfrak{C}((0,1))$ and $\mathcal{C}: (0,1) \to (0,1)$ be the mappings defined by

$$\mathbb{D}(p) = \begin{cases} \left\{\frac{1}{2}\right\} & \text{if } p \leq \frac{1}{2} \\ \left[\frac{1}{2}, \frac{3}{4}\right] & \text{if } p > \frac{1}{2} \end{cases} \quad and \quad \mathcal{C}(p) = p^2 \;\; \forall\;\; p \in (0,1).$$

Then clearly, $\mathcal{C}$ is onto and for every asymptotically regular sequence $\{\mathcal{C}p_n\}$ in $(0,1)$ with $\mathcal{C}p_{n+1} \in \mathbb{D}p_n$, $\mathcal{C}p_n \in \left[\frac{1}{2}, \frac{3}{4}\right]$ $\forall\;\; n$. Thus there exists a subsequence $\{\mathcal{C}p_{n_k}\}$ of $\{\mathcal{C}p_n\}$ such that $\lim_{n\to\infty} |\mathcal{C}p_{n_k} - l| = 0$, for some $l \in (0,1)$. Hence it follows that

$$\lim_{n\to\infty} \sigma(\mathcal{C}p_{n_k}, l) = \lim_{n\to\infty} |\mathcal{C}pn_k - l|e^t = \theta,$$

which implies $(\mathbb{D}, \mathcal{C})$ is $\mathbb{D}_{\mathcal{C}}$-weakly demicompact. Let $\phi: P \to P$ be a mapping defined by $\phi(t) = t$, then $\int_\theta^{\mathcal{H}(\mathbb{D}p,\mathbb{D}q)} \phi d_P = \theta$, as $\mathcal{H}(\mathbb{D}p, \mathbb{D}q) = \theta \;\; \forall \;\; p, q \in (0,1)$. $\psi\left(\int_\theta^{\mathcal{H}(\mathbb{D}p,\mathbb{D}q)} \phi d_P\right) = \theta \;\forall\, \psi \in \Psi$. Since $\theta \preceq \sigma(\mathcal{C}q, r) \;\forall\, p, q \in (0,1)$ and $r \in \mathbb{D}p$, we have $\theta \preceq \int_\theta^{\sigma(\mathcal{C}q,r)} \phi(t)dt \;\forall p, q \in (0,1)$ and $r \in \mathbb{D}p$.

Now it follows that, for any $\psi \in \Psi$ and $\lambda \in (0,1)$,

$$\theta \preceq \lambda\psi\left(\int_\theta^{\sigma(\mathcal{C}q,r)} \phi(t)dt\right) \;\forall\, p, q \in (0,1) \; and \; r \in \mathbb{D}p.$$

Thus by Theorem 3.3, $\mathbb{D}$ and $\mathcal{C}$ has a coincidence point in $(0,1)$.

Here note that if we replace $\mathcal{C}$ as

$$\mathcal{C}(p) = \begin{cases} p & \text{if } p \in \left(0, \frac{1}{4}\right) \\ \frac{p+2}{3} & \text{if } p \in [\frac{1}{4}, 1) \end{cases}$$

in the above example, then all the premises in Theorem 3.3, except the function $\mathcal{C}$ to be onto holds whereas its inference do not hold.

In follow we give an example to prove the necessity of a pair of mapping $(\mathbb{D}, \mathcal{C})$ to be $\mathbb{D}_{\mathcal{C}}$-weakly demicompact in Theorem 3.3.

Example 3.5 *Let* $S = \left\{0, \sum_{k=1}^{n} \frac{1}{k} : n = 1,2,3 \cdots\right\}$. *Let* E, P, d *and* $\mathcal{H}$ *be as in example 3.3. Let* $\mathbb{D}: S \to \mathfrak{C}(S)$ *and* $\mathcal{C}: S \to S$ *be the mappings defined by*

$$\mathbb{D}(p) = \begin{cases} \left\{0, \frac{3}{2}\right\} & \text{if } p = 0 \\ \left\{1, \frac{3}{2}\right\} & \text{if } p = 1 \\ \left\{0, 1, \sum_{k=1}^{n-1} \frac{1}{k}\right\} & \text{if } p = \sum_{k=1}^{n} \frac{1}{k}, n = 2,3,4, \cdots \end{cases}$$

and

$$\mathcal{C}(p) = \begin{cases} 1 & \text{if} p = 0 \\ 0 & \text{if} p = 1 \\ p & \text{if} p = \sum_{k=1}^{n} \frac{1}{k}, n = 2,3,4,\cdots \end{cases}$$

Then clearly $\mathcal{H}(\mathbb{D}p, \mathbb{D}q) = \theta$ and $\mathcal{C}$ is onto. Let $(p_n) = \left(\sum_{k=1}^{n} \frac{1}{k}\right)$ be a sequence, then $\{\mathcal{C}p_n\}$ is an asymptotically regular such that $\mathcal{C}p_n \in \mathbb{D}p_{n+1}$. Clearly $\{\mathcal{C}p_n\}$ is an increasing sequence which is not bounded above and hence no subsequence of $\{\mathcal{C}p_n\}$ is convergent. Thus the pair of mapping $(\mathbb{D}, \mathcal{C})$ is not $\mathbb{D}_{\mathcal{C}}$-weakly demicompact. But for any $\psi \in \Psi$ and $\phi: P \to P$, we have $\psi\left(\int_{\theta}^{\mathcal{H}(\mathbb{D}p,\mathbb{D}q)}\right) = \theta$ and since $\theta \preceq \sigma(\mathcal{C}q, r)\ \forall\, p, q \in S$ and $r \in \mathbb{D}p$, we have
$\theta \preceq \int_{\theta}^{\sigma(\mathcal{C}q,r)} \phi(t)dt\ \ \forall\, p, q \in S$ and $r \in \mathbb{D}p$. Thus for any $\psi \in \Psi$ and $\lambda \in (0,1)$,

$$\theta \preceq \lambda\psi\left(\int_{\theta}^{\sigma(\mathcal{C}q,r)} \phi(t)dt\right) \forall\, p, q \in S \; and \; r \in \mathbb{D}p.$$

Thus all the requirements in the hypothesis of Theorem 3.3 except the pair of mappings $(\mathbb{D}, \mathcal{C})$ to be $\mathbb{D}_{\mathcal{C}}$-weakly demicompact are satisfied and clearly its conclusion fails.

Theorem 3.6 *In Theorem 3.3 assume that* $(\mathbb{D}, \mathcal{C})$ *is weakly compatible and* $\mathcal{C}\mathcal{C}p = \mathcal{C}p$ *for each coincidence point* p *of* $\mathbb{D}$ *and* $\mathcal{C}$*, then* $\mathbb{D}$ *and* $\mathcal{C}$ *have a common fixed point.*
Proof. By Theorem 3.3, $\mathbb{D}$ and $\mathcal{C}$ have a point $r \in S$ such that $u = \mathcal{C}r \in \mathbb{D}r$ and hence
$\mathcal{C}r \in \mathcal{C}(\mathbb{D}r)$. As $(\mathbb{D}, \mathcal{C})$ is weakly compatible, we have $\mathcal{C}(\mathbb{D}r) = \mathbb{D}(\mathcal{C}r)$ and therefore
$\mathcal{C}u \in \mathbb{D}(\mathcal{C}r) = \mathbb{D}u$. But since $r \in S$, we have $\mathcal{C}u = \mathcal{C}\mathcal{C}r = \mathcal{C}r = u$ as desired.

Corollary 3.7 *Let* $(\mathbb{D}, \mathcal{C})$ *be a pair of* $\mathbb{D}_{\mathcal{C}}$*-weakly demicompact mappings such that* $\mathcal{C}$ *is onto. Suppose there exists a* $\mathcal{H}$*-cone metric on* $\mathcal{C}(S)$ *and* $\lambda \in (0,1)$ *such that*

$$\int_{\theta}^{\mathcal{H}(\mathbb{D}p,\mathbb{D}q)} \phi d_P \preceq \lambda \int_{\theta}^{\sigma(\mathcal{C}q,r)} \phi d_P, \forall\ p, q \in S \text{ and } r \in \mathbb{D}p. \tag{2}$$

Further if $(\mathbb{D}, \mathcal{C})$ is weakly compatible and $\mathcal{C}\mathcal{C}p = \mathcal{C}p$ for each coincidence point p of $\mathbb{D}$ and $\mathcal{C}$, then $\mathbb{D}$ and $\mathcal{C}$ have a common fixed point in S.
Proof. The proof follows trivially if we let $\psi(p) = p$ in Theorem 3.6.

Corollary 3.8 *Let* $\mathbb{D}: S \to \mathcal{C}(S)$ *be a weakly demicompact set valued mapping. If there exists* $\mathcal{H}$*-cone metric on* $\mathcal{C}(S)$ *and* $\lambda \in (0,1)$ *such that*

$$\int_{\theta}^{\mathcal{H}(\mathbb{D}p,\mathbb{D}q)} \phi d_P \preceq \lambda \int_{\theta}^{\sigma(q,r)} \phi d_P,\ \forall\ p, q \in S \text{ and } r \in \mathbb{D}p, \tag{3}$$

then $\mathbb{D}$ have a fixed point in S.
Proof. If we let $\mathcal{C}(p) = p$, then the proof follows from Corollary 3.7.

4 Application

Here, we use Corollary 3.8 to prove the existence of solutions for a set of Fredholm type integral inclusions. First let us fix some notations. Let $\mathcal{B} = \{f: [a_1, a_2] \to \mathbb{R} \mid f \text{ is continuous}\}$ and $\mathbb{C}_B$ be the collection of all nonempty closed and bounded subsets of $\mathcal{B}$.

Let $G = \{g_1, g_2, g_3, \cdots, g_n\}$. Then clearly G be an element of $\mathbb{C}_B$. Let $\mathbb{G}: [a_1, a_2] \to \mathbb{C}_B$ be a map defined by $\mathbb{G}(t) = G$. Let $m: [a_1, a_2] \times \mathbb{R} \to \mathbb{R}$ be a map that satisfies $\left|\int_{a_1}^{a_2} m(s, x(s))ds\right| < a_2 - a_1$ and

$$|m(s, x(s))g_i(t) - m(s, y(s))g_j(t)| \leq |y(s) - u(s)| \max\{|g_i(t)|, |g_j(t)|\} \quad (4)$$

for all $t \in [a_1, a_2]$ and $1 \leq i, j \leq n$, where $u(t) \in f(t) + \int_{a_1}^{a_2} m(s, x(s))G(t)ds$.

Let $M_i = \sup_{a_1 \leq t \leq a_2} g_i(t)$ and $M = \max_{1 \leq i \leq n} M_i$. Let $f(t) \in \mathcal{B}$ and

$$x(t) \in f(t) + \int_{a_1}^{a_2} m(s, x(s))G(t)ds, \quad t \in [a_1, a_2] \quad (5)$$

be a Fredholm type integral inclusion.

Let $S = span\{f(t), g_1(t), g_2(t), g_3(t), \cdots, g_n(t)\}$ and E be the set of all real valued continuous functions on $[0,1]$. Let

$$P = \{\mu(t): 0 \leq \mu(t) \ \forall\ t \in [0,1]\}$$

and Let $\sigma: S^2 \to E$ be a mapping defined by

$$\sigma(f, g) = \left(\sup_{t \in [a,b]} |f(t) - g(t)|\right) e^t \ \forall\ f, g \in S.$$

Then by considering E as a real Banach space with the normal cone P, it is easy to see that σ is a cone metric on S. Now we prove a theorem which gives a sufficient condition for the existence of a solution of (5).

Theorem 4.1 *If* $(a_2 - a_1)M < 1$*, then the Fredholm type integral inclusion* (5) *has atleast one solution in* S.

Proof. Let $\mathbb{D}: S \to \mathcal{C}(S)$ be defined by

$$\mathbb{D}x(t) = \left\{u(t) \in S \mid u(t) \in f(t) + \int_a^b m(s, x(s))G(t)ds\right\}.$$

Let $\{x_n(t)\}$ be an asymptotically regular sequence in S such that $x_n(t) \in \mathbb{D}x_{n+1}(t)$ and let $r_n = \left|\int_{a_1}^{a_2} m(s, x_n(s))ds\right|$. Then clearly the sequence $\{r_n\} \in \mathbb{R}$ is bounded and hence there must exist a convergent subsequence $\{r_{n_k}\}$ which converges to l(say). Now consider

$$\sigma(x_{n_k-1}(t), f(t) + Ml) = \left(\sup_{t \in [a_1, a_2]} \left|f(t) + \int_{a_1}^{a_2} m(s, x_n(s))g_i ds - f(t) - Ml\right|\right) e^t$$

Then $\lim_{k \to \infty} \sigma(x_{n_k-1}(t), f(t) + Ml) = \theta$ and hence $\mathbb{D}$ is weakly demicompact. Here note that to prove the existence of a solution for the integral inclusion (5), it is enough to show that $\mathbb{D}$ has a fixed point. Let $\mathcal{H}: \mathfrak{C}(S)^2 \to E$ be a mapping defined by

$$\mathcal{H}(\mathbb{D}x, \mathbb{D}y) = H_u(\mathbb{D}x, \mathbb{D}y)e^t,$$

where $\mathcal{H}_u$ is the usual Hausdorff metric. Let $v(t) \in \mathbb{D}x(t)$ and $z(t) \in \mathbb{D}y(t)$, then $v(t) = f(t) + \int_{a_1}^{a_2} m(s, x(s))g_i(t)ds$ and $z(t) = f(t) + \int_{a_1}^{a_2} m(s, y(s))g_j(t)ds$ for some i, j. Now for all $t \in [a_1, a_2]$,

$$|v(t) - z(t)| = \left|\int_{a_1}^{a_2} m(s, x(s))g_i ds - \int_{a_1}^{a_2} m(s, y(s))g_j ds\right|$$

$$\leq \int_{a_1}^{a_2} |m(s, x(s))g_i - m(s, y(s))g_j| ds$$

$$\leq (a_2 - a_1)|y(s) - u(s)|\max\{|g_i(t)|, |g_j(t)|\}$$
$$\leq \sup_{a_1 \leq s \leq a_2} |y(s) - u(s)|(a_2 - a_1)M$$

Thus it follows that $\sigma(v(t), z(t)) \leq (a_2 - a_1)\sigma(y(t), u(t))M$. Similarly, we can prove that $\mathcal{H}(x(t), y(t)) \leq (a_2 - a_1)\sigma(y(t), u(t))M$ where $u(t) \in \mathbb{D}x(t)$. Now if we let $\phi(x) = x$ in (3), then $\mathcal{H}(x, y) \leq \lambda\sigma(y, u)$ where $u \in \mathbb{D}x$. Thus it is easy to see that $\mathbb{D}$ has a fixed point by using $(a_2 - a_1)M < 1$ and Corollary 3.8.

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